Chapter 1 Homework

Homework #1.2+1.3 – Math 676

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#Math 676/Chapter 1

Homework Assignment:

- **Section 1.2:** Exercise 4.
- **Section 1.3:** Exercises 7 and 8.
- **Proof revision exercise:** Learn the proof of Lemma 1.3.16 and rewrite it without looking. Then answer the following questions:
  - What omissions/mistakes you have?
  - Are there parts of the proof that are not clear?
  - What would you have written differently?

Section 1.2

4) 4. Solve the following linear equations with complex coefficients.

a) $2(5z - 2i) = 10(z - 3) + 30 - 4i$ Distribute

$10z - 4i = 10z - 30 + 30 - 4i$ CLT

$10z - 4i = 10z - 4i$ → Subtract $(10z - 4i)$ from both sides

$0 = 0$

Linear equation is an identity and z is any complex number

b) $(2 + i)z + (3 - i) = 3z - (2 - 6i)$

$2z + iz + 3 - i = 3z - 2 + 6i$

$-3z \quad -3 + i \quad -3z \quad -3 \quad +i$

$(-1 + i)z = -5 + 7i$

$$\frac{(-1+i)z}{(-1+i)} = \frac{-5+7i}{(-1+i)}$$

$$z = \frac{-5+7i}{-1+i} \cdot \frac{-1-i}{-1-i} = \frac{5 + 5i - 7i \overset{-7(-1)}{- 7i^2}}{(-1)^2 + (i)^2} = \frac{5 - 2i + 7}{1+1} = \frac{12-2i}{2}$$

$$z = \frac{12-2i}{2} = 6 - i$$

$z = 6 - i$

Linear equation is conditional and $z = 6 - i$

C) $4z + \frac{5}{2}i = (1+3i)z + \frac{1}{2}$

$4z + \frac{5}{2}i = 1z + 3iz + \frac{1}{2}$

$-3iz \quad -1z \quad -\frac{5}{2}i \qquad -1z \quad -3iz \qquad -\frac{5}{2}i$

$3z - 3iz = \frac{1}{2} - \frac{5}{2}i$

$\frac{(3-3i)z}{3-3i} = \frac{\frac{1-5i}{2}}{3-3i}$ ← Multiply by Reciprocal! It'll be easier to not use that formula.

$z = \frac{1-5i}{2} \cdot \frac{1}{3-3i}$

$z = \frac{1-5i}{6-6i} \cdot \frac{(6+6i)}{(6+6i)} = \frac{6+6i-30i-30i^2}{36+36} = \frac{36-24i}{72} = \frac{1}{2} - \frac{1}{3}i$ $\quad -30(-1)$

$z = \frac{1}{2} - \frac{1}{3}i$

Linear equation is conditional and $z = \frac{1}{2} - \frac{1}{3}i$

Section 1.3

7) 7. Solve each equation among complex numbers.

Note. Show all your steps. Your final answer should be a set of solutions, which could be a finite set, the empty set $\emptyset$, or the whole set of complex numbers $\mathbb{C}$. If you get a non-empty finite set, $\boxed{\textbf{state}}$ if your solutions are rational, real irrational, or non-real complex.

(a) $4z^2 + 9 = 0$

$-9 \qquad -9$

$\frac{4z^2}{4} = \frac{-9}{4}$

$\sqrt{z^2} = \sqrt{\frac{-9}{4}}$

$z = \pm\sqrt{\frac{-9}{4}}$

$z = \pm\sqrt{\frac{9}{4}}\,i$

$z = \pm\frac{3}{2}i$

So, $z = \left\{ \begin{array}{l} \frac{3}{2}i \\ -\frac{3}{2}i \end{array} \right.$ These solutions are imaginary, non-real complex.

(b) $4z^2 + 4z + 5 = 0$

$4^2 - 4(4)(5)$
$\hookrightarrow$ negative $\rightarrow$ complex solutions

$a = 4 \quad b = 4 \quad c = 5$

$$z = \frac{-4 \pm \sqrt{(4)^2 - 4(4)(5)}}{2(4)} = \frac{-4 \pm \sqrt{16-80}}{8} = \frac{-4 \pm \sqrt{-64}}{8} = \frac{-4 \pm \sqrt{64}\,i}{8} = \frac{-4 \pm 8i}{8} = \frac{-1 \pm 2i}{2}$$

$$z = \begin{cases} -\frac{1}{2} + i \\ -\frac{1}{2} - i \end{cases}$$ These solutions are non-real complex.

(c) $4z^2 + 24z + 36 = 0$

$24^2 - 4(4)(36)$
$576 - 576 = 0 \nearrow$ one real solution

$$\frac{4z^2 + 24z + 36}{4} = \frac{0}{4}$$

$z^2 + 6z + 9 = 0$

$\sqrt{(z+3)^2} = \sqrt{0}$

$z + 3 = 0$

$z = -3$

$z = -3$ (with multiplicity 2). This solution is rational.

(d) $2z^2 + 4 = 0$

$$\frac{2z^2 + 4}{2} = \frac{0}{2}$$

$z^2 + 2 = 0$

$z^2 = -2$

$z = \pm\sqrt{-2}$

$z = \pm\sqrt{2}\,i$

$$z = \begin{cases} \sqrt{2}\,i \\ -\sqrt{2}\,i \end{cases}$$ These solutions are imaginary, non-real complex.

(e) $\sqrt{z^2} = \sqrt{-5 + 12i}$

$a = -5 \quad b = 12$

$|z| = \sqrt{(-5)^2 + (12)^2} = \sqrt{25 + 144} = \sqrt{169} = 13$

$$\pm\sqrt{z} = \pm\left(\sqrt{\frac{|z| + a}{2}} + \text{sign}(b)\sqrt{\frac{|z| - a}{2}}\, i\right)$$

$z = \pm\sqrt{-5 + 12i}$

$$z = \pm\left(\sqrt{\frac{13-5}{2}} + \sqrt{\frac{13+5}{2}}\, i\right)$$

$$z = \pm\left(\sqrt{\frac{8}{2}} + \sqrt{\frac{18}{2}}\, i\right)$$

$z = \pm(\sqrt{4} + \sqrt{9}\,i) = \pm(2 + 3i)$

$$z = \begin{cases} 2 + 3i \\ -2 - 3i \end{cases}$$ These solutions are non-real complex.

(f) $(2+i)z^2 + (3-2i)z - (48-12i) = 0$

$a = 2+i \qquad b = 3-2i \qquad c = -48+12i$

$(3-2i)(3-2i)$

$(3-2i)^2 = 3^2 - 2(6i) + (-2i)^2$
$9 - 12i + 4i^2$
$9 - 12i - 4 = 5 - 12i$

$-4(2+i)(-48+12i)$
$(-8-4i)(-48+12i) = 384 - 96i + 192i - 48i^2$
$= 384 + 96i + 48$
$= 432 + 96i$

$$z = \frac{-(3-2i) \pm \sqrt{(3-2i)^2 - 4(2+i)(-48+12i)}}{2(2+i)}$$

$$z = \frac{-3+2i \pm \sqrt{5-12i + 432 + 96i}}{4+2i}$$

$$z = \frac{-3+2i \pm \sqrt{437+84i}}{4+2i}$$

$\sqrt{437+84i}$: $|437+84i| = \sqrt{437^2 + 84^2}$
$= \sqrt{190969 + 7056}$
$= \sqrt{198025}$
$= 445$

$\pm\sqrt{z} = \pm\left(\sqrt{\frac{|z|+a}{2}} + \text{sign}(b)\sqrt{\frac{|z|-a}{2}}\, i\right)$

$\pm\left(\sqrt{\frac{445+437}{2}} + \sqrt{\frac{445-437}{2}}\, i\right)$

$\pm\left(\sqrt{\frac{882}{2}} + \sqrt{\frac{8}{2}}\, i\right)$

$\pm\left(\sqrt{441} + \sqrt{4}\, i\right)$

$\pm(21 + 2i)$

$$z = \frac{-3+2i \pm (21+2i)}{4+2i}$$

$$z_1 = \frac{-3+2i+21+2i}{4+2i} = \frac{(18+4i)^{\div 2}}{(4+2i)^{\div 2}} = \frac{9+2i}{2+i} \cdot \frac{(2-i)}{(2-i)} = \frac{18-9i+4i+2}{4+1} = \frac{20-5i}{5} = 4-i$$

$$z_2 = \frac{-3+2i-21-2i}{4+2i} = \frac{(-24)^{\div 2}}{(4+2i)^{\div 2}} = \frac{-12}{2+i} \cdot \frac{(2-i)}{(2-i)} = \frac{-24+12i}{4+1} = \frac{-24+12i}{5} = \frac{-24}{5} + \frac{12}{5} i$$

$$z = \begin{cases} 4-i \\ \frac{-24}{5} + \frac{12}{5} i \end{cases}$$

These solutions are non-real complex.

8) Determine two complex numbers whose sum is -1 and whose product is 1.

$s = -1 \qquad d = 1$

$t^2 - (-1)t + 1 = 0$

$t^2 + 1t + 1 = 0$

$$t = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2}$$

$$t = \begin{cases} \frac{-1}{2} + \frac{\sqrt{3}}{2} i \\ \frac{-1}{2} - \frac{\sqrt{3}}{2} i \end{cases}$$

So, the two complex numbers are $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$ and $\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)$

Proof revision exercise: Learn the proof of Lemma 1.3.16 and rewrite it without looking. Then answer the following questions:

- What omissions/mistakes you have?
- Are there parts of the proof that are not clear?
- What would you have written differently?

Lemma 1.3.16. Let α and β be two complex numbers such that their sum is s and their product is d. Then α and β are the solutions of the polynomial equation:

$$t^2 - st + d = 0.$$

Pf: Let $s = \alpha + \beta$ and $d = \alpha\beta$. Using Substitution, $t^2 - st + d = 0$ becomes

$$t^2 - (\alpha+\beta)t + \alpha\beta = 0$$
$$t^2 - \alpha t - \beta t + \alpha\beta = 0$$
$$t(t-\alpha) - \beta(t+\alpha) = 0$$
$$(t-\alpha)(t-\beta) = 0$$

Thus, α and β are Solutions to $t^2 - st + d$ ∎

- With my proof vs. the proof in the notes, I showed the "straightforward Computation" in Order to be more clear. I don't believe that I had any Omissions, aside from calling out the fact that I was factoring, which I would write differently next time.

Homework #1.4+1.5 – Math 676

[redacted] #Math 676/Chapter 1 [redacted]

Homework Assignment:

- **Section 1.4:** Exercises 5 and 6.
- **Section 1.5:** Exercises 1 and 2.

Section 1.4

5) 5. Factor the polynomial $P(x) = \frac{1}{2}x^4 - x^3 + \frac{1}{2}x^2 - 2x + 2$ into irreducible terms with rational coefficients by following the steps below.

(a) Determine a polynomial $Q(x)$ with all integer coefficients and having the same zeros of $P(X)$.

(b) Apply Theorem 1.4.9 in order to determine all rational zeros.

(c) Determine all linear factors of $P(x)$.

(d) Apply the long division (or synthetic division) to one linear factor at a time in order to factor $P(x)$ completely.

a) To find $Q(x)$, we will multiply $P(x)$ by 2 to clear the denominators

$$2\cdot(P(x)) = \left(\frac{1}{2}x^4 - x^3 + \frac{1}{2}x^2 - 2x + 2\right)\cdot 2 \qquad P(x) = \frac{1}{2}Q(x)$$

$$Q(x) = x^4 - 2x^3 + x^2 - 4x + 4$$

b) For Theorem 1.4.9 we know: Possible rational zeros of $P(x) = \dfrac{\text{Divisors of the constant term}}{\text{Divisors of the leading coefficient}}$

Divisors of Constant: $\pm1, \pm2, \pm4$ So, Possible rational zeros of $Q(x) = \dfrac{\pm1, \pm2, \pm4}{\pm1} = \pm1, \pm2, \pm4$

LeadinCoeff: ±1

C+D) Try $x=1$: $\frac{Q(x)}{(x-1)}$

1	1	-2	1	-4	4
	↓	1	-1	0	-4
	1	-1	0	-4	0 ✓

So, $Q(x) = (x^3 - x^2 - 4)(x-1)$

$x=1$ Nope

$x=-1$ Nope

Try $x=2$ $\frac{Q(x)}{(x-2)}$

2	1	-1	0	-4
	↓	2	2	4
	1	1	2	0 ✓

So, $Q(x) = (x^2+x+2)(x-1)(x-2)$

This won't reduce any further, would need QF.

Because we can't factor any further with rational roots,

$$P(x) = \frac{1}{2}(x^2+x+2)(x-1)(x-2)$$

6) 6. Let $P(x)$ be a monic polynomial with all integer coefficients. Prove that any rational root of $P(x)$ must be an integer.

LC of 1

Pf: Let $P(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ with all integer coefficents.

Let $\frac{p}{q}$ be any rational root of $P(x)$. So, we know $P(\frac{p}{q}) = 0$. we are trying to show that $\frac{p}{q} \in \mathbb{Z}$.

If we substitute $x = (\frac{p}{q})$,

$$P\left(\frac{p}{q}\right) = \left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \dots + a_1\left(\frac{p}{q}\right) + a_0$$

$$q^n \cdot 0 = \left(\frac{p^n}{q^n} + a_{n-1}\frac{p^{n-1}}{q^{n-1}} + \dots + a_1 \cdot \frac{p}{q} + a_0\right) \cdot q^n$$

$$0 = p^n + a_{n-1}p^{n-1} \cdot q + a_{n-2}p^{n-2}q^2 + \dots + a_1 p q^{n-1} + a_0 q^n$$

$$0 = p^n + q\left(a_{n-1}p^{n-1} + a_{n-2}p^{n-2}q + \dots + a_1 p q^{n-1} + a_0 q^{n-1}\right)$$

$$p^n = -q\left(a_{n-1}p^{n-1} + a_{n-2}p^{n-2}q + \dots + a_1 p q^{n-1} + a_0 q^{n-1}\right)$$

Thus, p^n is divisible by q.

However, by Thm 1.4.9, $\frac{p}{q}$ is reduced to lowest terms. So, since q divides both p^n and p, q must be either 1 or -1, because q cannot have prime factors.

Thus, $\frac{p}{q}$ must be an integer, and so any rational root is an integer of $P(x)$. ∎

Theorem 1.4.9. Rational Zero Theorem.
Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with integer coefficients. We call a_0 the $\boxed{\textbf{constant term}}$ and a_n the $\boxed{\text{leading coefficient}}$.

Let $\frac{p}{q}$ be a rational zero of $P(x)$, that is to say, $P\left(\frac{p}{q}\right) = 0$ and the fraction $\frac{p}{q}$ is reduced to lowest terms. Then p is a divisor of the constant term a_0 and q is a divisor of the leading coefficient a_n.

Section 1.5

1) 1. Write the following complex numbers in polar form, then determine the power corresponding to the given value of n.

(a) $z = 1 + i$, $n = 3$

$$r = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$$

$\tan(\theta) = 1$

$$\theta = \frac{\pi}{4}$$

$$z = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)\right)$$

$$z^3 = (\sqrt{2})^3\left(\cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right)\right)$$

$$z^3 = 2\sqrt{2}\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

$$z^3 = -2 + 2i$$

(b) $z = 3i,\ n = 5$

$r = \sqrt{0^2+3^2} = \sqrt{9} = 3$

$\theta = \frac{\pi}{2}$

$Z = 3\left(\cos\left(\frac{\pi}{2}\right) + i\ \sin\left(\frac{\pi}{2}\right)\right)$

$Z^5 = 3^5\left(\cos\left(\frac{5\pi}{2}\right) + i\ \sin\left(\frac{5\pi}{2}\right)\right)$

$= 243\left(\cos\left(\frac{\pi}{2}\right) + i\ \sin\left(\frac{\pi}{2}\right)\right)$

$= 243\left(0 + i(1)\right)$

$Z^5 = 243i$

Thinking from 0 to 2π, $\frac{5\pi}{2} = 2\pi + \frac{\pi}{2}$

(c) $z = 1 - \sqrt{3}i,\ n = 4$

$r = \sqrt{(1)^2+(\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$

$\tan(\theta) = \frac{-\sqrt{3}}{1} \rightarrow \theta = 2\pi + \left(-\frac{\pi}{3}\right) = \frac{5\pi}{3}$

$Z = 2\left(\cos\left(\frac{5\pi}{3}\right) + i\ \sin\left(\frac{5\pi}{3}\right)\right)$

$Z^4 = 2^4\left(\cos\left(\frac{20\pi}{3}\right) + i\ \sin\left(\frac{20\pi}{3}\right)\right)$

$= 16\left(\cos\left(\frac{2\pi}{3}\right) + i\ \sin\left(\frac{2\pi}{3}\right)\right)$

$= 16\left(-\frac{1}{2} + i\ \frac{\sqrt{3}}{2}\right)$

$Z^4 = -8 + 8\sqrt{3}\ i$

$\frac{20\pi}{3} = \frac{2\pi}{3}$

(d) $z = -2i,\ n = 21$

$r = \sqrt{0^2+(-2)^2} = \sqrt{4} = 2$ $\quad \theta = \frac{3\pi}{2}$

$-2i$

$Z = 2\left(\cos\left(\frac{3\pi}{2}\right) + i\ \sin\left(\frac{3\pi}{2}\right)\right)$

$Z^{21} = 2^{21}\left(\cos\left(\frac{63\pi}{2}\right) + i\ \sin\left(\frac{63\pi}{2}\right)\right)$

$= 2^{21}\left(\cos\left(\frac{3\pi}{2}\right) + i\ \sin\left(\frac{3\pi}{2}\right)\right)$

$= 2^{21}\left(0 + i(-1)\right)$

$Z^{21} = -2097152i$

$\frac{63\pi}{2} = \frac{3\pi}{2}$

(e) $z = -1 - i,\ n = 4$

$r = \sqrt{(-1)^2+(-1)^2} = \sqrt{1+1} = \sqrt{2}$

$\tan(\theta) = \frac{-1}{-1} = \tan\theta = 1 \rightarrow \frac{\pi}{4}$ in 3rd quad

$\theta = \frac{5\pi}{4}$

$Z = \sqrt{2}\left(\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right)$

$Z^4 = (\sqrt{2})^4\left(\cos\left(\frac{20\pi}{4}\right) + i\sin\left(\frac{20\pi}{4}\right)\right)$

$= 4\left(\cos(5\pi) + i\sin(5\pi)\right)$

$= 4\left(\cos(\pi) + i\sin(\pi)\right)$

$= 4\left((-1) + i(0)\right)$

$Z^4 = -4$

$5\pi = \pi$

2) 2. Determine the cubic roots of the following complex numbers.

(a) $z = 1 - i$

$r = \sqrt{1^2+(-1)^2} = \sqrt{2}$

$\theta = \frac{7\pi}{4}$

$Z = \sqrt{2}\left(\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right)$

$\theta + 2\pi k$

$k=0$: $\frac{7\pi}{4} + 2\pi(0) = \frac{7\pi}{4} \Rightarrow \frac{\frac{7\pi}{4}i}{3} = \frac{7\pi i}{12}$

$k=1$ $\quad \frac{7\pi}{4} + \frac{8\pi}{4} = \frac{15\pi}{4} \Rightarrow \frac{15\pi i}{4}\cdot\frac{1}{3} = \frac{15\pi i}{12}$

$k=2$ $\quad \frac{7\pi}{4} + \frac{16\pi}{4} = \frac{23\pi}{4} \Rightarrow \frac{23\pi i}{4}\cdot\frac{1}{3} = \frac{23\pi i}{12}$

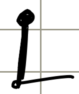
$\left(2^{1/2}\right)^{1/3} = 2^{1/6}$

The cube roots are: $2^{1/6}\cdot e^{\frac{7\pi i}{12}}$

$2^{1/6}\cdot e^{\frac{15\pi i}{12}}$

$2^{1/6}\cdot e^{\frac{23\pi i}{12}}$

(b) $z = 27i$



$r = \sqrt{0^2 + 27^2} = 27$

$\theta = \frac{\pi}{2}$

$z = 27\left(\cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right)\right)$

$\sqrt[3]{27} = 3$

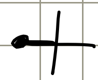
$k=0: \frac{\pi}{2} \Rightarrow \frac{\pi i}{2} \cdot \frac{1}{3} = \frac{\pi i}{6}$

$k=1: \frac{\pi}{2} + \frac{4\pi}{2} = \frac{5\pi}{2} \Rightarrow \frac{5\pi i}{6}$

$k=2: \frac{\pi}{2} + \frac{8\pi}{2} = \frac{9\pi}{2} \Rightarrow \frac{9\pi i}{6}$

The cube roots are: $3 \cdot e^{\pi i/6}$, $3 \cdot e^{5\pi i/6}$, $3 \cdot e^{9\pi i/6}$

(c) $z = -1$



$r = \sqrt{(-1)^2 + 0^2} = \sqrt{1} = 1$

$\theta = \pi$

$z = 1(\cos(\pi) + i \sin(\pi))$

$\sqrt[3]{1} = 1$

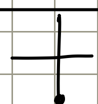
$k=0: \pi \Rightarrow \frac{\pi i}{3}$

$k=1: \pi + 2\pi \Rightarrow \frac{3\pi i}{3} = \pi i$

$k=2: \pi + 4\pi \Rightarrow \frac{5\pi i}{3}$

The cube roots are: $e^{\pi i/3}$, $e^{\pi i}$, $e^{5\pi i/3}$

(d) $z = -i$



$r = \sqrt{0^2 + (-1)^2} = 1$

$\theta = \frac{3\pi}{2}$

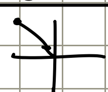
$k=0: \frac{3\pi}{2} \Rightarrow \frac{3\pi i}{6} = \frac{\pi i}{2}$

$k=1: \frac{3\pi}{2} + \frac{4\pi}{2} \Rightarrow \frac{7\pi i}{6}$

$k=2: \frac{3\pi}{2} + \frac{8\pi}{2} \Rightarrow \frac{11\pi i}{6}$

The cube roots are: $e^{\pi i/2}$, $e^{7\pi i/6}$, $e^{11\pi i/6}$

(e) $z = -4\sqrt{2} + 4\sqrt{2}i$



$4\pi = \frac{16\pi}{4}$

$r = \sqrt{(-4\sqrt{2})^2 + (4\sqrt{2})^2} = \sqrt{16 \cdot 2 + 16 \cdot 2}$

$= \sqrt{64} = 8$

$\tan\theta = \frac{4\sqrt{2}}{-4\sqrt{2}} = -1$

$\tan\theta = -1 \xrightarrow{\frac{\pi}{4}} \theta = \frac{3\pi}{4}$

$z = 8\left(\cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right)\right)$

$\sqrt[3]{8} = 2$

$k=0: \frac{3\pi}{4} \Rightarrow \frac{3\pi i}{12} = \frac{\pi i}{4}$

$k=1: \frac{3\pi}{4} + \frac{8\pi}{4} \Rightarrow \frac{11\pi i}{12}$

$k=2: \frac{3\pi}{4} + \frac{16\pi}{4} \Rightarrow \frac{19\pi i}{12}$

The cube roots are: $2e^{\pi i/4}$, $2e^{11\pi i/12}$, $2e^{19\pi i/12}$

Homework #1.6 - Math 676

#Math 676/Chapter 1

Homework Assignment: Section 1.6: Exercise 1.

Section 1.6

1) 1. Determine all the solutions of the following cubic equations, then determine an approximated value to 6 decimal digits by using a calculator. Are you able to determine whether the solutions are rational, real irrational, or non-real complex?

(a) $x^3 + 5x + 3 = 0$;

✓ Depressed Cubic

Rational Roots $\pm 1 \pm 3 \to$ None Work

$p = 5 \quad q = 3$

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

$$x = \sqrt[3]{\frac{-3}{2} + \sqrt{\frac{9}{4} + \frac{125}{27}}} + \sqrt[3]{\frac{-3}{2} - \sqrt{\frac{9}{4} + \frac{125}{27}}} \approx -0.564$$

$$uv = -\frac{5}{3} \implies \frac{-p^3}{27} = \frac{-125}{27}$$

$$u^3 + v^3 = -q = -3$$

$$t^2 + qt - \frac{p^3}{27} \quad > \quad t^2 + 3t - \frac{125}{27} = 0$$

$$t = \frac{-3 \pm \sqrt{9 - 4\left(-\frac{125}{27}\right)}}{2} = \frac{-3 \pm \sqrt{\frac{743}{27}}}{2}$$

* Simplified w/ Maple

$$t_1 = u^3 = \frac{-3}{2} + \frac{\sqrt{2229}}{18}$$

$$u = \sqrt[3]{\frac{-3}{2} + \frac{\sqrt{2229}}{18}}$$

$$t_2 = v^3 = \frac{-3}{2} - \frac{\sqrt{2229}}{18}$$

$$v = \sqrt[3]{\frac{-3}{2} - \frac{\sqrt{2229}}{18}}$$

Checked $u \cdot v$ in maple, got $-\frac{5}{3}$ ✓

$$x_1 = u + v = \sqrt[3]{\frac{-3}{2} + \frac{\sqrt{2229}}{18}} + \sqrt[3]{\frac{-3}{2} - \frac{\sqrt{2229}}{18}} \approx -0.564 \text{ (real irrational)}$$

$$x_2 = \omega u + \bar{\omega} v = \sqrt[3]{\frac{-3}{2} + \frac{\sqrt{2229}}{18}}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) + \sqrt[3]{\frac{-3}{2} - \frac{\sqrt{2229}}{18}}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) \approx 0.282050 + 2.288811 i$$

(Non-real Complex)

$$x_2 = \bar{\omega} u + \omega v = \sqrt[3]{\frac{-3}{2} + \frac{\sqrt{2229}}{18}}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) + \sqrt[3]{\frac{-3}{2} - \frac{\sqrt{2229}}{18}}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) \approx 0.282050 - 2.288811 i$$

(b) $x^3 + 7x^2 + 4x - 1 = 0.$

Possible Rational Roots: ± 1

Neither work

Let $x = y - \frac{7}{3}$.

Using Maple, we get $y^3 - \frac{37}{3}y + \frac{407}{27} = 0$ So, $p = \frac{-37}{3}$ $q = \frac{407}{27}$

$uv = \frac{-p}{3} = \frac{37}{9} \Rightarrow (uv)^3 = \frac{50653}{729}$ ✓

$u^3 + v^3 = \frac{-407}{27}$ ✓

$t^2 + \frac{407}{27}t + \frac{50653}{729} = 0$

$729t^2 + 10989t + 50653 = 0$

$t = \frac{-10989 \pm \sqrt{(10989)^2 - 4(729)(+50653)}}{2(729)}$

$y = u + v$

$-3uv = -\frac{37}{3}$ $uv = \frac{37}{9}$ ✓ $u^3v^3 =$

$-(u^3 + v^3) = \frac{407}{27}$

Using Maple

$t = \frac{-407}{54} \pm \frac{37\sqrt{269}}{54}$

$r = \sqrt{a^2 + b^2} = \frac{37\sqrt{37}}{27} e^{\theta i}$

$\theta = \pi + \tan^{-1}\left(\frac{\frac{37\sqrt{3}}{18}}{\frac{407}{54}}\right)$

$\theta = \pi - \tan^{-1}\left(\frac{3\sqrt{3}}{11}\right)$

↳ doesn't come out nice.

$u^3 = -\frac{407}{54} + \frac{37i\sqrt{3}}{18} = \frac{37\sqrt{37}}{27} \cdot e^{\theta i}$

$u = \sqrt[3]{-\frac{407}{54} + \frac{37i\sqrt{3}}{18}} = \frac{\sqrt{37}}{3} \cdot e^{\frac{1}{3}\theta i}$

$v^3 = -\frac{407}{54} - \frac{37i\sqrt{3}}{18} = \frac{37\sqrt{37}}{27} \cdot e^{\theta i}$

$v = \sqrt[3]{-\frac{407}{54} - \frac{37i\sqrt{3}}{18}} = \frac{\sqrt{37}}{3} e^{\frac{\theta}{3} i}$

$\theta = \pi + \tan^{-1}\left(\frac{3\sqrt{3}}{11}\right)$

Checked $u \cdot v = \frac{37}{9}$ using maple and it does!

$y_1 = u + v = \sqrt[3]{-\frac{407}{54} + \frac{37i\sqrt{3}}{18}} + \sqrt[3]{-\frac{407}{54} - \frac{37i\sqrt{3}}{18}} \approx 2.520434$

$x_1 = y_1 - \frac{7}{3} \approx 0.187101$ (real irrational).

$y_2 = \omega u + \bar{\omega} v = \sqrt[3]{-\frac{407}{54} + \frac{37i\sqrt{3}}{18}}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \sqrt[3]{-\frac{407}{54} - \frac{37i\sqrt{3}}{18}}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \approx -4.011379$

$x_2 = y_2 - \frac{7}{3} \approx -6.344712$ (real irrational)

$y_3 = \bar{\omega} u + \omega v = \sqrt[3]{-\frac{407}{54} + \frac{37i\sqrt{3}}{18}}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) + \sqrt[3]{-\frac{407}{54} - \frac{37i\sqrt{3}}{18}}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \approx 1.490945$

$x_3 = y_3 - \frac{7}{3} \approx -0.842388$ (real irrational)

Homework #1.7 - Math 676

#Math 676/Chapter 1

Homework Assignment: Section 1.7: Exercises 1, 2.

Section 1.7

1) 1. Show that we get the same answer if we choose $y = -2$ as a solution of the Ferrari resolvent in Example 1.7.2.

$a=2,\ b=2,\ c=6,\ d=3$

If $y=-2$, then, $s = \frac{-2}{2} = -1$

$$\beta = \sqrt{2s+b+\frac{a^2}{4}} = \sqrt{-2+2+\frac{2^2}{4}} = \sqrt{1} = 1$$

$$\gamma = \frac{c+as}{2\sqrt{2s+b+\frac{a^2}{4}}} = \frac{6+(-2)}{2\sqrt{1}} = \frac{4}{2} = 2$$

Now Solve:

$$x^2 + \left(\frac{a}{2}+\beta\right)x + (s+\gamma) = x^2 + \left(\frac{2}{2}+1\right)x + (-1+2) = x^2+2x+1=0$$

$$(x+1)^2 = 0$$

$$x=-1,\ x=-1$$

$$x^2 + \left(\frac{a}{2}-\beta\right)x + (s-\gamma) = x^2 + \left(\frac{2}{2}-1\right)x + (-1-2) = x^2+0x-3=0$$

$$x^2=3$$

$$x=\pm\sqrt{3}$$

So, we can factor $x^4+2x^3-2x^2-6x-3 = (x+\sqrt{3})(x-\sqrt{3})(x+1)^2$, and have shown we get the same result. //

2)

2. Solve the equation

$$x^4 + 2x^3 - 9x^2 - 2x + 8 = 0$$

by applying the algorithm explained in this section.

Hint. You may try to factor the Ferrari resolvent by grouping.

Forgot this hint, but ended up @ the same spot!!

$$x^4 + 2x^3 - 9x^2 - 2x + 8 = 0$$

So $a = 2,\ b = 9,\ c = 2,\ d = -8$

Now, Solve the Ferrari resolvent

$$y^3 + by^2 + (4d - ac)y - (c^2 - 4bd - a^2 d) = 0$$

$$y^3 + 9y^2 - 36y - 324 = 0$$

$4d - ac = 4(-8) - 2(2) = -32 - 4 = -36$

$c^2 - 4bd - a^2 d = 2^2 - 4(9)(-8) - 2^2(-8) = 4 + 288 + 32 = 324$

Possible Rational Roots

$\cancel{\pm 1}, \cancel{\pm 2}, \cancel{\pm 3}, \cancel{\pm 4}, \pm 6, \pm 9, \pm 12, \pm 18, \pm 27, \pm 36, \pm 54, \pm 81, \pm 108, \pm 162, \pm 324$

* Used Maple to check to see if any existed.

$y = 6$ is a rational root!

	1	9	-36	-324
6	↓	6	90	324
	1	15	54	0

So, $y^3 + 9y^2 - 36y - 324 = (y-6)(y^2 + 15y + 54)$
$= (y-6)(y+9)(y+6)$

So, solving the Ferari resolvent,

$y = 6,\ y = -6,\ y = -9$

Chose $y = 6$, so $s = \frac{6}{2} = 3$

$$\beta = \sqrt{2(3) + 9 + \frac{2^2}{4}} = \sqrt{6 + 9 + 1} = \sqrt{16} = 4$$

$$\gamma = \frac{2 + 2(3)}{2(4)} = \frac{8}{8} = 1$$

$a = 2$

$b = 9$

$c = 2$

$d = -8$

Now Solve

$$x^2 + \left(\tfrac{a}{2} + \beta\right)x + (s + \gamma) = x^2 + \left(\tfrac{2}{2} + 4\right)x + (3+1) = x^2 + 5x + 4 = 0$$

$(x+1)(x+4) = 0$

$x = -1 \quad x = -4$

$$x^2 + \left(\tfrac{a}{2} - \beta\right)x + (s - \gamma) = x^2 + \left(\tfrac{2}{2} - 4\right)x + (3-1) = x^2 - 3x + 2 = 0$$

$(x-1)(x-2) = 0$

$x = 1 \quad x = 2$

So, we can factor $x^4 + 2x^3 - 9x^2 - 2x + 8 = (x+1)(x+4)(x-1)(x-2)$.