Group Theory Exam - Math 676

Ma[illegible]ehn #Math 676

Part I:

(I) Choose 3 of the following exercises. General remark: unless specified otherwise, there is no need to justify as long as your answer is correct and complete.

(a) Determine all the subgroups of S_3. $S_3 = \{e, (12), (13), (23), (123), (132)\}$

$\hookrightarrow |S_3| = 6 \rightarrow$ Orders of subgroups must be either order 1, 2, 3, or 6.

Theorem 2.4.6. Finite Subgroup Test Let G be a group and let H be a $\boxed{\textbf{nonempty}}$ finite subset of G. If H is closed under the operation of G, then H is a subgroup of G.

order 1: $H_1 = \{e\}$

order 2:
$H_2 = \{e, (12)\} \rightarrow (12)(12) = e$
$H_3 = \{e, (13)\} \rightarrow (13)(13) = e$
$H_4 = \{e, (23)\} \rightarrow (23)(23) = e$
Closed so Subgroup

order 3: $H_5 = \{e, (123), (132)\} = A_3$

$(123)(123) = (132)$ Add. ($1\to2\to3$, $2\to3\to1$, $3\to1\to2$)
$(132)(132) = (123)$ ($1\to3\to2$, $2\to1\to3$, $3\to2\to1$)
$(123)(132) = e$ ($1\to3\to1$, $2\to1\to2$, $3\to2\to3$)
$(132)(123) = e$ ($1\to2\to1$, $2\to3\to2$, $3\to1\to3$)
No more order 3, adding any more transpositions creates S_3.

order 6: $H_6 = \{e, (12), (13), (132), (123), (23)\} = S_3$

$(12)(13) = (132)$ ($1\to3$, $2\to1$, $3\to1\to2$)
$(13)(12) = (123)$ ($1\to2$, $2\to1\to3$, $3\to1$)
$(12)(132) = (13)$ ($1\to3\to3$, $2\to1\to2$, $3\to2\to1$)
$(13)(132) = (23)$ ($1\to3\to1$, $2\to1\to3$, $3\to2\to2$)

(d) Determine all the subgroups of $\mathbb{Z}_{15}$. $= \{0, 1, 2, 3, \dots, 14\}$

$\hookrightarrow |\mathbb{Z}_{15}| = 15 \rightarrow$ Orders of subgroups must be either order 1, 3, 5, 15.

order 1: $H_1 = \{0\}$
order 3: $H_2 = \{0, 5, 10\} = \langle 5 \rangle$
order 5: $H_3 = \{0, 3, 6, 9, 12\} = \langle 3 \rangle$
order 15: $H_4 = \mathbb{Z}_{15} = \{0, 1, 2, 3, \dots, 14\}$

(e) Determine the inner product of $<(1,2)>$ and $<(3,4)>$ in S_4.

$\langle(12)\rangle\langle(34)\rangle$ $\quad \langle(12)\rangle = \{e, (12)\}$ $\quad \langle(34)\rangle = \{e, (34)\}$

We recall that the internal product of two subsets S and T of a group G is the set

$ST := \{x \in G | \, x = st$ for some element s is S and some element t in $T\}$.

	e	(34)
e	e	(34)
(12)	(12)	(12)(34)

$\langle(12)\rangle\langle(34)\rangle = \{e, (12), (34), (12)(34)\}$

Part II:

(II) Choose 3 of the following exercises. General remark: unless specified otherwise, you **need** to justify your answer.

(B)

Look for a proof that the group A_5 is simple, then sketch the main steps of the proof and point out which steps are unclear. Mention the source where you found the proof.

Proof found at

www.math.brown.edu/dabrown/MA/f1516/251/AnSimple_PresentationNotes.pdf

Proposition 1. The group A_5 is simple.

Proof. There are 5 conjugacy classes of A_5. The table below contains a representative and the order of each one:

Representative from Conjugacy Class	(1)	(12345)	(21345)	(12)(34)	(123)
Order of Conjugacy Class	1	12	12	15	20

Any normal subgroup $N \triangleleft A_5$ must be a union of these conjugacy classes, including (1). Further, the order of N would divide the order A_5. However the only divisors of $|A_5| = 60$ that are possible by adding up 1 and any combination of $\{12, 12, 15, 20\}$ are 60 and 1. Thus $N = \{1\}$ or $N = A_n$, and A_n is simple. □

① The Proof begins by mentioning that their are 5 conjugacy classes of A_5. It shows a representative and the order of each Conjugacy Class:
- $|(1)| = 1$
- $|(12345)| = 12$
- $|(21345)| = 12$
- $|(12)(34)| = 15$
- $|(123)| = 20$

② Next, it mentions that any Normal subgroup of A_5 must include $(1) = e$, and must be a union of those Conjugacy Classes.

③ However, the only divisors of $|A_5| = 60$ that can be created by adding 1 and any of the other orders is 1 and 60.

So N must be $\{e\}$ or A_5.

One of the first steps that was unclear was how the conjugacy classes were crafted. I believe that adding this step could help with making the proof more clear.

Additionally, the second step in the proof, discussing why N needs to be a union of Conjugacy classes could also help with understanding.

(E)

Let H be a normal subgroup of order 3 in a group G of order 111. Prove that G/H is abelian.

Theorem 2.6.15. Let G be a finite group of order p where p is a prime number. Then G is cyclic; that is, G is isomorphic to $\mathbb{Z}_p$.

Proposition 2.3.22. Let $\phi : G \to H$ be a group isomorphism, in particular, the groups G and H are isomorphic. Then the group G is abelian if and only if the group H is abelian.

$|H| = 3, \quad |G| = 111$

Since $|H| = 3$, and $|G| = 111$, $|G/H| = \frac{|G|}{|H|} = \frac{111}{3} = 37$. Since $|G/H| = 37$, and we know that 37 is a prime number, then we know that $|G/H|$ is cyclic, and is isomorphic to $\mathbb{Z}_{37}$. Since $\mathbb{Z}_{37}$ is abelian, it follows that $|G/H|$ is abelian. ■

(F) Let G be a group of order 10 containing exactly 5 elements of order 2. Let H be the set of the 5 elements of G not of order 2. Prove that H is a normal subgroup of G.

$g^2=e$

Pf: Prove H is a subgroup of G.

Identity: Since $|e| \neq 2$, that means that $e \in H$, so H has the identity element.

Closure: Let $a, b \in H$. We want to show that $ab \in H$.

Suppose $|ab| = 2$, so $ab \notin H$. This means that $(ab)^2 = e$.
So $abab = e$, this means that $ab = b^{-1}a^{-1}$ $\quad ab \cdot b^{-1}a^{-1} = aea^{-1} = aa^{-1} = e$
However, that is a contradiction, since $|a| \neq 2$ and $a \neq a^{-1}$, so $ab \in H$.

Inverses: Let $a \in H$. Show $a^{-1} \in H$.

Since the elements of order 2 are their own inverses, if $|a^{-1}| = 2$, $a = a^{-1}$, but then $a \notin H$.
However, since $a \in H$, $a^{-1} \in H$.

So H is a subgroup of G. //

Prove H is normal.

To show that H is normal, we need to show that $\forall g \in G$, $gHg^{-1} = H$.

Since H is closed, all elements in H will satisfy the property gHg^{-1}.

If $g \notin H$, $g = g^{-1}$. so $gHg^{-1} = gHg$. We also know that $g^2 = e$.

We know that $gHg^{-1} \subseteq H$. Substitution

$gHg \subseteq H$ Multiply by g on both left and right.

$g(gHg)g \subseteq gHg$
$ggHgg \subseteq gHg$
$eHe \subseteq gHg$
$H \subseteq gHg$
$H \subseteq gHg^{-1}$, So $gHg^{-1} = H$, so H is normal subgroup of G. ∎ //