Homework #1 - Math 511

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1) **4 characters**

a) How many possible if each taken from n-set?

↳ for this, order matters, and we will allow repeats.

So, n^4.

b) Smallest n that guarentees at least $1{,}000{,}000{,}000 = 10^9$?

↳ $n^4 \geq 10^9$

$n \geq (10^9)^{1/4}$

$n \geq 10^{9/4}$

$n \geq 177.82 \approx 178.$

So, the smallest n can be is 178.

8 characters

↳ Similarly

n^8.

↳ $n^8 \geq 10^9$

$n \geq (10^9)^{1/8}$

$n \geq (10)^{9/8}$

$n \geq 13.335 \approx 14$

So the smallest n can be is 14

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2) How many subsets of [20].....

a) smallest element 4, largest 15?

↳ Because we are constraing the elements, we are left w/ $\{4, 5, 6, \ldots 15\}$, or 12 elements to work with. We know that each subset must contain 4 and 15, so the smallest the subset can be is 2 elements, and the largest is all 12 elements.

$$\text{Total Subsets} = \underset{\substack{2\text{ elements}\\ \{4,15\}}}{\underline{1}} + \underset{3\text{ elem.}}{\underline{\binom{10}{1}}} + \underset{4\text{ elem}}{\underline{\binom{10}{2}}} + \cdots + \underset{11\text{ elems.}}{\underline{\binom{10}{9}}} + \underset{\substack{12\text{ elem.}\\ \{4,5,6,\ldots 15\}}}{\underline{1}} = 1 + 10 + 45 + 120 + 210 + 252 + 210 + 120 + 45 + 10 + 1$$

$$= 1024$$

$$\underset{\{4\}}{\underline{1}} \cdot \underline{\binom{10}{1}} \cdot \underset{\{15\}}{\underline{1}}$$

Choose 1 of everything left

I also could have solved this by computing $2^{10} \rightarrow$ finding all subsets of $\{5, 6, \ldots, 14\}$ since 4 and 15 are fixed. Thought of this after writing everything else out ☺.

b) Contains no even elements?

↳ Constraing the elements, we are left w/ $S = \{1, 3, 5, 7, \ldots 19\}$, and $|S| = 10$. Thus, the number of all subsets can be computed by $2^{|S|} = 2^{10} = 1{,}024$

c) have size 10 and no #'s larger than 17?

↳ we have 17 elements to pick from ($\{1, 2, 3, \ldots 17\}$), and want to choose 10 of them where order doesn't matter w/o repeats.

So, the # of subsets are $\binom{17}{10} = 19{,}448$

3) (5 points) Page 15 #17 Find the number of 3-lists of the form (x_1, x_2, x_3), where each x_i is a nonnegative integer and $x_1 + x_2 + x_3 = 10$.

↳ We want to have 10-multisets, taken from [3].

For example: $| \underbrace{0\ 0\ 0\ 0}_{x_1 = 4} | \underbrace{0\ 0\ 0}_{x_2 = 3} | \underbrace{0\ 0\ 0}_{x_3 = 3}$

So, we can calculate this using $\left(\!\binom{3}{10}\!\right) = \binom{10+3-1}{10} = \binom{12}{10} = 66$

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4) (5 points) Page 23 #8 Let k and n be positive integers satisfying $k < n$. How many subsets of $[n]$ are not also subsets of $[k]$?

We know that $[k] = \{1, 2, 3, \dots k\}$ and $[n] = \{1, 2, 3, \dots k, k+1, \dots n\}$.
We also know that $[k]$ has 2^k subsets and $[n]$ has 2^n subsets. Because we don't want any subsets that appear in $[k]$, there must be at least 1 value from $\{k, k+1, \dots n\}$

So, Total Subsets from $[n]$ = Subsets of $[k]$ + Subsets not in $[k]$

$$\Rightarrow 2^n = 2^k + |\text{Subsets not in } [k]|$$

$$2^n - 2^k = |\text{Subsets not in } [k]|$$

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5) Shidoku: To solve this problem, I drew a lot of pictures ☺.
Let a, b, c, d stand for digits {1,2,3,4}

Step 1: fill in NW corner

a	b		
c	d		

There are 4! ways that you could fill this in

Step 2: Fill in Rest of top Row

a	b	c	d
c	d		

There are 2 ways to fill this in: cd or dc

Step 3: Fill in Rest of Col 1.

a	b	c	d
c	d		
b			
d			

Two ways to fill this in $\begin{smallmatrix} b \\ d \end{smallmatrix}$ or $\begin{smallmatrix} d \\ b \end{smallmatrix}$

Assuming I fix these values, I currently have $4! \cdot 2 \cdot 2 = 96$ boards.

From here, I had to look at specific options:

Option 1: a, b in row 2

a	b	c	d
c	d	a	b
b		d	_
d		b	_

2 possible solutions $\begin{smallmatrix} c \\ a \end{smallmatrix}$ or $\begin{smallmatrix} a \\ c \end{smallmatrix}$

Option 2: b, a in row 2

a	b	c	d
c	d	b	a
b	a	d	c
d	c	a	b

only 1 solution

So, using the fixed values, there are now 3 more ways to fill in the board.

Thus, the total # of Shidoku boards is:

$$4! \cdot 2 \cdot 2 \cdot 3 = 288$$

6) (5 points) Page 23 #17 Find the number of 4-lists of the form (x_1, x_2, x_3, x_4) where each x_i is a nonnegative integer and $x_1 + x_2 + x_3 + 4x_4 = 15$.

If

$x_4 = 0, \quad x_1 + x_2 + x_3 = 15 \Rightarrow \left(\!\binom{3}{15}\!\right)$

$x_4 = 1, \quad x_1 + x_2 + x_3 = 11 \Rightarrow \left(\!\binom{3}{11}\!\right)$

$x_4 = 2, \quad x_1 + x_2 + x_3 = 7 \Rightarrow \left(\!\binom{3}{7}\!\right)$

$x_4 = 3, \quad x_1 + x_2 + x_3 = 3 \Rightarrow \left(\!\binom{3}{3}\!\right)$

Adding these together:

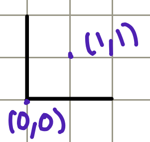
$$\left(\!\binom{3}{15}\!\right) + \left(\!\binom{3}{11}\!\right) + \left(\!\binom{3}{7}\!\right) + \left(\!\binom{3}{3}\!\right)$$

$$= \binom{17}{15} + \binom{13}{11} + \binom{9}{7} + \binom{5}{3}$$

$$= 136 + 78 + 36 + 10$$

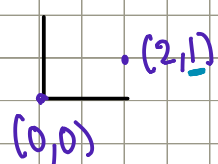
$$= 260$$

7) Dorothy Problem:



It would take her 2 steps.

$\left.\begin{array}{l} N, E \text{ or } E, N \end{array}\right\}$ 2 ways to do it

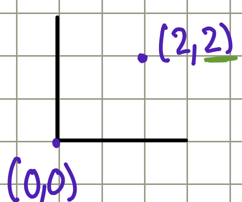


It would take 3 steps:

$\left.\begin{array}{l} NEE \\ ENE \\ \text{or } EEN \end{array}\right\}$ 3 ways to do it.

1 step N, choose when to use it!

$\binom{3}{1} = 3$



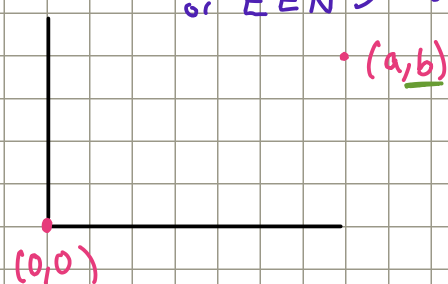
It would take her 4 steps:

$\left.\begin{array}{l} NNEE \\ NENE \\ NEEN \\ ENNE \\ ENEN \\ EENN \end{array}\right\}$ 6 ways

2 steps N

Choose when!

$\binom{4}{2} = 6$



It would take a+b steps

Choose when to go b steps N!

In General: Dorothy's Steps have $\binom{a+b}{b}$ options!

8) (10 points) Page 32 #6 Give a bijective proof: The number of n-digit binary numbers with exactly k ones equals the number of k-subsets of $[n]$.

Let A = set of all n-digit binary #'s with k 1's and
B = set of all k-subsets of $[n]$.

Let $f: A \to B$ given by $f(a_1 a_2 a_3 \ldots a_n) = \{i \in [n] \mid a_i = 1\}$

Well Defined:

Let $a \in A$. Thus, a is a n-digit binary # with exactly k 1's. Let $i_1, i_2, \cdots i_k$ be the positions of 1's in a, where $1 \le i_1 < i_2 < \ldots < i_k \le n$. Now, $f(a) = S$, where $S = \{i_1, i_2, \ldots, i_k\} \subseteq B$. Since there are exactly k 1s in a, this means that $|S| = k$.

Injective:

Let $a_1, a_2 \in A$. Assume $f(a_1) = f(a_2)$. Because the sets of positions where a_1 and a_2 have 1's are identical, this means the positions of 1's in a_1 and a_2 are identical. Thus $a_1 = a_2$.

Surjective:

Let $T=\{i_1,i_2,\dots,i_k\}\in B$, thus T is a k-subset of $[n]$. Then let S be a binary # w/exactly k 1's such that

$S=a_1a_2\dots a_n$ where $a_i=\begin{cases}1 & \text{if } i\in T\\ 0 & \text{if } i\notin T\end{cases}$. Using our function, $f(S)=T$, since $a_i=1$ for each $i\in B$. So f is surjective

Thus, f is bijective, so $|A|=|B|$ ■ !!.

9) (10 points) Page 32 #12 Let $\mathcal{E}$ and $\mathcal{O}$ be the set of even- and odd-sized subsets of $[n]$, respectively. If n is odd then the set complement function maps sets in $\mathcal{E}$ to sets in $\mathcal{O}$. Is this a bijection? Prove or disprove.

Pf: Let E and Θ be defined as above, + let $n=2m+1$, for $m\in\mathbb{N}$.

Let $f: E\to\Theta$ given by $f(S)=S^c$

Well-defined:

Let A be an even sized subset of $[n]$. So $A\in E$, so $|A|=2k$, for some $k\in\mathbb{N}$. We know that A^c is a subset of $[n]$ with size $|A^c|=n-|A|$. Thus, $|A^c|=2m+1-2k$, $=2(m-k)+1$, which is odd. So $A^c\in\Theta$.

Injective:

Let $A_1, A_2\in E$. Assume $f(A_1)=f(A_2)$. Thus $A_1^c=A_2^c$. By taking the complement again, $(A_1^c)^c=(A_2^c)^c$, proves $A_1=A_2$. Thus f is injective.

Subjective:

Let $T\in\Theta$. Let $S=T^c$. Since $|T^c|=n-|T|$, which is odd, $|T^c|$ must be in E. Thus, $|S|\in E$, so $T\in$ mg f.

So, f is a bijection. ■

10) (10 points) Page 32 #13 This exercise outlines a bijective proof of the formula $\left(\!\binom{n}{k}\!\right)=\binom{k+n-1}{k}$ from Section 1.1. Let A be the set of k-multisets taken from $[n]$ and let B be the set of k-subsets of $[k+n-1]$. Assume that the k-multiset $\{a_1,a_2,\dots,a_k\}$ is written in nondecreasing order: $a_1\le a_2\le\cdots\le a_k$. Define $f: A\to B$ by

$$f(\{a_1,a_2,\dots,a_k\})=\{a_1,a_2+1,a_3+2,\dots,a_k+k-1\}.$$

This function, and proof, is originally due to Euler.

a) Prove that the outputs of f are indeed k-subsets of $[k+n-1]$. That is, prove that f is well-defined.

Let $S\in A$. Thus, S is a k-multiset of $[n]$ $S=\{a_1,a_2,\dots,a_k\}$ and S is written in nondecreasing order. We need to show that $f(S)$ is a k-subset of $[k+n-1]$. By our function, $f(S)=\{a_1,a_2+1,a_3+2,\dots a_k+k-1\}$. Because S is in increasing order, this means a_k is the largest element. If $a_k+k-1\in[k+n-1]$, f is well-defined. The maximum $a_k=n$, so $f(\{n\})=\{n+k-1\}$, and $\{n+k-1\}\in[k+n-1]$. So, the function is well-defined.

b) Prove that f is a bijection.

Injective:

Let $S, T \in A$. Thus, they are k-multisets of $[n]$, written in nondecreasing order.
Let $S = \{a_1, a_2, \ldots, a_k\}$ and $T = \{b_1, b_2, \ldots, b_k\}$. Assume $f(S) = f(T)$.
So, $\{a_1, a_2+1, a_3+2, \ldots, a_k+k-1\} = \{b_1, b_2+1, b_3+2, b_k+k-1\}$. Since we have specified that S, T are in nondecreasing order, this means that:

$a_1 = b_1$, $a_2+1 = b_2+1$, $a_3+2 = b_3+2, \ldots, a_k+k-1 = b_k+k-1$. This simplifies to:
$a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$, $\ldots$, $a_k = b_k$.

So, $S = T$, proving f is injective.

Need to show $f(S) = T$

Surjective:

Let $T \in B$. So, T is a k-subset of $[k+n-1]$, let $T = \{a_1, a_2, a_3, \ldots, a_k\}$, s.t. $a_1 < a_2 < \ldots < a_k$

Let $S = \{a_1, a_2-1, a_3-2, \ldots, a_k-k+1\}$. We know that $S \in A$ because since $a_k \leq k+n-1$, Then

$$\begin{array}{r} a_k \leq k+n-1 \\ -k+1 \quad -k \quad +1 \\ \hline a_k-k+1 \leq n \end{array}$$

so $S \in A$.

So, $f(S) = T$, since $f(\{a_1, a_2-1, a_3-2, \ldots a_k-k+1\}) = \{a_1, (a_2-1)+1, (a_3-2)+2, \ldots, (a_k-k+1)+k-1\}$
$= \{a_1, a_2, a_3, \ldots, a_k\}$.

Thus, f is surjective.

Because f is both inj. and surjective, f is a bijection.

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11) (8 points) A degree n monomial in variables $x_1, x_2, \ldots, x_k$ is a product of the form $x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$, where $n_1, n_2, \ldots,$ and n_k are non-negative integers and $n_1 + n_2 + \cdots + n_k = n$. How many degree n monomials in variables $x_1, x_2, \ldots, x_k$ are there?

Using examples:

$n = 3$, $k = 2$

looking for # of monomials $x_1^{n_1} x_2^{n_2}$ when $n_1 + n_2 = 3$.

Using $n_1 + n_2 = 3$, we are looking for a 3 multiset taken from a 2-set. $\Rightarrow \left(\!\binom{2}{3}\!\right) = \binom{3+2-1}{3} = \binom{4}{3} = 4$

So, $0+3=3 \longrightarrow x_1^0 x_2^3$
$1+2=3 \longrightarrow x_1^1 x_2^2$
$2+1=3 \longrightarrow x_1^2 x_2^1$
$3+0=0 \longrightarrow x_1^3 x_2^0$

So there are 4 degree 3 monomials in 2 variables.

In general, using the idea of multisets, we can find $\left(\!\binom{k}{n}\!\right) = \binom{k+n-1}{n}$ degree n monomials in k-variables.

12) (10 points) Suppose there are $2n$ balls in a bag, where a are red and b are blue, and $a + b = 2n$. One ball is removed at random and then replaced. Then another ball is removed at random. What is the probability that the balls are of different color? Show that this probability is maximized when $a = b = n$.

$P(\text{Red}) = \frac{a}{2n}$ $\quad P(\text{Blue}) = \frac{b}{2n}$

$$P(\text{Different Color}) = P(R)\cdot P(B) + P(B)\cdot P(R)$$
$$= \frac{a}{2n}\cdot\frac{b}{2n} + \frac{b}{2n}\cdot\frac{a}{2n}$$
$$= \frac{ab}{4n^2} + \frac{ba}{4n^2}$$
$$= \frac{2ab}{4n^2}$$

$$P(\text{Different Color}) = \frac{ab}{2n^2}$$

Maximized when $a = b = n$: $\quad a + b = 2n$, $\quad a = 2n - b$

$$\frac{ab}{2\left(\frac{a+b}{2}\right)^2} =$$

$$P = \frac{(2n-b)b}{2n^2}$$
$$P = \frac{2nb - b^2}{2n^2}$$

Take Deriv w.r.t. b

$$P' = \frac{2n^2(2n-2b)}{4n^4} = \frac{2n-2b}{2n^2} = \frac{n-b}{n^2}$$

To find the maximum, Set $= 0$

$$\frac{n-b}{n^2} = 0$$
$$n - b = 0$$
$$n = b$$

Because $a + b = 2n$, $\quad a + n = 2n$, $\quad a = n$

So, $a = n = b$

Because $a = n = b$, $P(\text{Different Color}) = \frac{ab}{2n^2} = \frac{a^2}{2a^2} = \frac{1}{2}$

I had to look up some things to remind me about optimization

Trying Something new to keep everything organized! :)
Let me know if writing is too small! Zoom gets deceptive on a tablet!