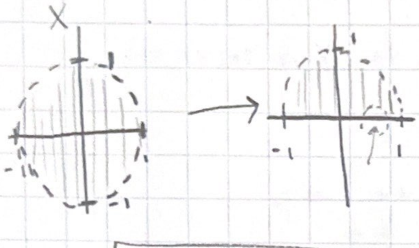
1) $f: \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x,y) = (x, |y|)$. **HW #10**

a) This is called the folding function, because it takes all negative y's and folds it up to the positive inteces.

b) Is f open? (For every U in X, $f(U)$ is open in Y).

Let $U = \{(x,y) \mid x,y \in \mathbb{R}, x^2+y^2<1\}$. We know U is open in $\mathbb{R}^2$



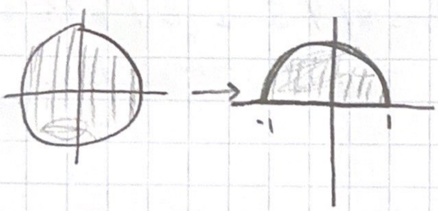
With the folding function, $f(U)$ will fold up in half.

This is no longer open, as points along the x-axis will have neighborhoods with points not in the set

$\boxed{f \text{ is not open}}$

c) Is f closed?

Let $V = \{(x,y) \mid x,y \in \mathbb{R}, x^2+y^2 \leq 1\}$



V is closed in $\mathbb{R}^2$.

$f(V)$ folds up in half.

$f(V)$ is closed because it includes all the boundry points.

It doesn't matter what V is, $f(V)$ will be closed.

$\boxed{f \text{ is Closed}}$

2) a) D is a dense subset of X + $f: X \to Y$ is continuous + surjective. So $f(D)$ is dense in Y.

Pf We know D is dense in X. Let V, be a closed subset of Y s.t. $f(D) \subseteq V$. Because f is continuous, $f^{-1}(V)$ is closed in X by Thrm 9.2.1. Because $f^{-1}(V) \ni D$, and D is dense in X, $f^{-1}(V) = X$. Because f is surjective $f(f^{-1}(V)) = f(X) = Y$ and $f(f^{-1}(V)) = V$.

Since $V = Y$, $f(D)$ is dense in Y.

2b) ~~-~~ X is seperable, $f: X \to Y$ is cont. + surjective

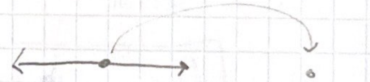
Prove Y is seperable.

Pf Let X be seperable. Thus, X must have a countable dense subset, A. Since A is dense in X, and f is continuous, + surjective, by part A, $f(A)$ is dense, and because A is countable, $f(A)$ is also countable. So, Y has a countably dense subset, so Y is seperable. ∎

3) If X is 2nd Countable, $f: X \to Y$ is cont., surjective + open, Then Y is 2nd Countable.

Pf Let X be 2nd countable. Thus, $\exists$ a countable Basis $\mathcal{B}$ s.t. $\{B_i \mid i \in I\}$ We need to show that Y has a countable Basis. Using Thrm. 10.1.4, We know that $\{f(B_i) \mid i \in I\}$, is a basis for Y. Since $\{B_i \mid i \in I\}$ is countable, $\{f(B_i) \mid i \in I\}$ is countable also. Thus Y is 2nd Countable. ∎

4) Y is $\mathbb{R}$ Finite complement Let $f: \mathbb{R}_{STD} \to Y$ by $f(x) = x$



Prove f is continuous. Show if open $U \in Y$ $f^{-1}(U)$ is open in $\mathbb{R}_{STD}$

Let U be an open set in Y. Since U is open, $U = \emptyset$ or $\mathbb{R} \setminus U$ is finite in $\mathbb{R}$. Show $f^{-1}(U)$ is open. We know $f^{-1}(U)$ is all $x \in \mathbb{R}$ s.t. $f(x) = U$.
If $U = \emptyset$, $f^{-1}(U) = \emptyset$ which is open in $\mathbb{R}_{STD}$. The other option is that $f(x) \in U$, then, $f^{-1}(U) = \mathbb{R}$ which is open in $\mathbb{R}_{STD}$. Thus f is continuous.
Now, we prove f is not a homeomorphism. We know f is not bijective, Because for a closed set in Y, it would map back to an open set in $\mathbb{R}$., So f^{-1} is not continuous. Thus f is continuous but not a homeomorphism ∎