Homework #12

1) Prove $\mathbb{R}_{LL}$ is not connected.

Pf: Notice that $[0,1)$ is open in $\mathbb{R}_{LL}$. Thus, $\mathbb{R}/[0,1) = (-\infty,0) \cup [1,\infty)$, which is the complement of $[0,1)$ is closed. But we know that $\mathbb{R}/[0,1)$ is also open, because it is a union of open sets. So, by Thrm. 12.1.2, $\mathbb{R}_{LL}$ is not connected. ■

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2) Prove X is connected iff X is not the union of two disjoint nonempty closed sets

($\Rightarrow$) Supposed X is connected. Assume by way of contradiction, that there exists closed, non-empty sets A, B s.t. $A \cap B = \emptyset$, and $X = A \cup B$. Since A is closed, X/A is open. Similarly, X/B is open. Since $X = A \cup B$, we know $X/A = B$ and $X/B = A$., so $(X/B) \cap (X/A) = \emptyset$, + $(X/B) \cup (X/A) = X$, which shows that A + B form a seperation of X, which contradicts the fact that X is connected.

($\Leftarrow$) Assume X is not the union of two disjoint non-empty closed sets, + prove X is connected.

Let X not be connected. Then, We know A, B are a separation of X, s.t. A, B are open and $X = A \cup B$ and $A \cap B = \emptyset$.

Since A + B are open, $X \setminus A$ + $X \setminus B$ are closed. As above, we know $X \setminus A = B$ + $X \setminus B = A$, so $(X \setminus A) \cup (X \setminus B) = X$. However, this contradicts are assumption that X is not the union of disjoint non-empty closed sets. Thus X is connected.

So We have shown that X is connected iff X is not the union of 2 disjoint non-empty closed sets. ■

3) Prove the set of Rationals as subspace of $\mathbb{R}_{STD}$ is not Connected.

Pf/ Let $\mathbb{Q}$ be the set of Rationals.

Let $a \in \mathbb{R}$, where a is a irrational number. We know that $(-\infty, a)$ + (a, ∞) are open in $\mathbb{R}_{STD}$. Let $A = (-\infty, a) \cap \mathbb{Q}$ is open in $\mathbb{Q}$ in the subspace topology. Similarly, $B = (a, \infty) \cap \mathbb{Q}$ is open. Since, $A \cap B = \emptyset$ and $A \cup B = \mathbb{Q}$, we've proven than A + B form a separation of $\mathbb{Q}$. Thus, $\mathbb{Q}$ is not connected.

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4. Prove that $\bar{S}$ is connected

Pf/ Let $\bar{S} = \{(x, \sin(\frac{1}{x})) \mid x > 0\} \cup \{(0,y) \mid -1 \leq y \leq 1\} = U \cup V$, so $U = \{(x, \sin(\frac{1}{x})) \mid x > 0\}$ and $V = \{(0,y) \mid -1 \leq y \leq 1\}$

Assume to reach a contradiction, that $\bar{S}$ is not connected, so then there is a seperation of $\bar{S}$. Let U, V be the seperation, so $U \cap V = \emptyset$, $U \cup V = \bar{S}$ and U, V are open.

It is clear that U, V are disjoint, and obviously, $U \cup V = \bar{S}$. However, V is closed, so thus, $\bar{S}$ is connected.

5.) Let $f: X \to Y$ be contin. + surjective. If X is connected, then Y is connected.

Pf/ Assume X is connected. Assume, by way of contradiction, that A, B are a separation of Y, where $A \cup B = Y$ and $A \cap B = \emptyset$, where A + B are both open. Since f is continuous, $f^{-1}(A)$ + $f^{-1}(B)$ are open in X. Also, $f^{-1}(A) \cap f^{-1}(B) = \emptyset$. Also, f is surjective $f^{-1}(A) \cup f^{-1}(B) = X$. All these facts show that X is seperated by $f^{-1}(A)$ and $f^{-1}(B)$ which is a contradiction.

Thus if X is connected, Y is connected when $f: X \to Y$ is contin. + surjective.