# Homework #13

1) $d: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by $d(\overset{A}{(x_1, y_1)}, \overset{B}{(x_2, y_2)}) = \max\{|x_2 - x_1|, |y_2 - y_1|\}$

a) ① (1.1) Prove $d(A,B) \geq 0$: This is obvious, because the Abs. Value of anything is $\geq 0$, so the max of two numbers both $\geq 0$ also must be $\geq 0$.

(1.2) Assume $d(A,B) = 0$: Show $A = B$. In order for $d((x_1,y_1),(x_2,y_2)) = 0$, then $|x_2 - x_1| = 0$ and $|y_2 - y_1| = 0$. If this is true, it implies that $x_2 = x_1$ and $y_2 = y_1$, so $(x_1, y_1) = (x_2, y_2)$.

② Symmetry → $d((x_1,y_1),(x_2,y_2)) = d((x_2,y_2),(x_1,y_1))$: We know that $|a-b| = |b-a|$. So, then it implies that $|x_2 - x_1| = |x_1 - x_2|$ + $|y_2 - y_1| = |y_1 - y_2|$. Since these are the same, the maximum val is the same.

thus $d((x_1,y_1),(x_2,y_2)) = d((x_2,y_2),(x_1,y_1))$ $\forall (x_1,y_1), (x_2,y_2) \in \mathbb{R}^2 \times \mathbb{R}^2$.

③ $\Delta$-ineq.: Let $X = (x_1, y_1)$, $Y = (x_2, y_2)$, $Z = (x_3, y_3)$, which are all in $\mathbb{R}^2 \times \mathbb{R}^2$.

By the $\Delta$-inequality in the reals,

Show: $d(x,z) \leq d(x,y) + d(y,z)$.

$$\max\{|x_3 - x_1|, |y_3 - y_1|\} \leq \max\{|x_1 - x_2| + |x_2 - x_3|, |y_1 - y_2| + |y_2 - y_3|\}$$

$$\Rightarrow \max\{|x_3 - x_1|, |y_3 - y_1|\} \leq \max\{|x_1 - x_2|, |y_1 - y_2|\} + \max\{|x_2 - x_3|, |y_2 - y_3|\}$$

$$\Rightarrow d(x,z) \leq d(x,y) + d(y,z) \quad \forall x, y, z \text{ in } \mathbb{R}^2 \times \mathbb{R}^2.$$

So, $d$ is a metric. :)

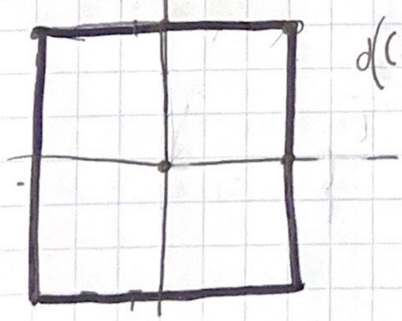
b) $B_d((0,0); 1)$

$d((1,1),(0,0)) = \max\{|1-0|, |1-0|\} = 1$ ✓

$= \quad 1 \quad , \quad 1$ ✓

$d((1,0),(0,0)) = \max\{|1-0|, |0-0|\} = 1$ ✓

$1 \quad 0$



2) Let $d_X: X \times X \to \mathbb{R}$ where $d_X$ is a metric. Let $Y \subseteq X$. Show $Y$ is a metric Space.

Pf Let $d_Y: Y \times Y \to \mathbb{R}$ by $d_Y(y_1, y_2) = d_X(y_1, y_2)$ $\forall y_1, y_2 \in Y$.

We know that $X$ is a metric Space. So, we know the following are true: $\forall x, y \in X$.

① $\forall_{x,y \in X}$, $d_X(x,y) \geq 0$, and if $d_X(x,y) = 0$, then $x = y$.

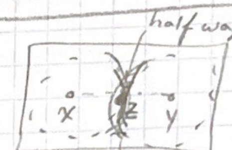
② $d_X(x,y) = d_X(y,x)$

③ $\forall x, y, z \in X$, $d_X(x,z) = d_X(x,y) + d_X(y,z)$

Because, $Y \subseteq X$, every element in $Y$ is also in $X$. Because $X$ is a metric space, Statements ①,②,③ are true for all $x, y, z \in X$, so they must also be true for all $y_1, y_2, y_3 \in Y$ since $y_1, y_2, y_3 \in X$.

Thus, $Y$ is a metric Space ■ :)

3) Prove every metric space is Hausdorff.




Pf Let $X$ be a Metric space. & let $x, y \in X$, & $x \neq y$.

Because $X$ is a metric space, $d(x,y) \geq 0$, so lets say $d(x,y) = \varepsilon$

To prove Hausdorff, show $\exists\, U, V \subseteq X$ s.t $x \in U$, $y \in V$ & $U \cap V = \emptyset$.

Let $U = B(x, \frac{\varepsilon}{2})$ & $V = B(y, \frac{\varepsilon}{2})$. Thus, $x \in U$ & $V \ni y$.

Assume $X$ is not Hausdorff. Thus, $\exists\, z \in U \cap V$, so $U \cap V \neq 0$. Because $z \in U$, $d(x,z) < \frac{\varepsilon}{2}$ & Similarly $z \in V$, so $d(y,z) < \frac{\varepsilon}{2}$.

Because $X$ is a metric space, the $\Delta$-inequality property must hold. But since both $d(x,z) + d(y,z) < \frac{\varepsilon}{2}$, then $d(x,z) + d(y,z) < \varepsilon$.

I think I got there legally & not magically :)

But, w/ the $\Delta$-ineq. prop of metric spaces

$d(x,y) \leq d(x,z) + d(y,z)$

so $\varepsilon \leq d(x, z + d(y,z) < \varepsilon$, so $\varepsilon < \varepsilon$ which is a contradiction!

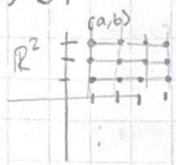
Thus All Metric spaces are Hausdorff. ■ :)

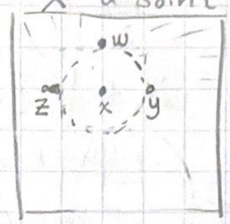
4. If Every uncountable Set in $X$ has a L.P, then $X$ is seperable

a) $D_1 \Rightarrow$ any 2 p.ts $x, y \in D_1$ satisfy $d(x,y) > 1$.

not 100% what this is, did I miss a def.?

i) Why is $D_1$ is a unit 1 mesh? I believe it is a unit 1 mesh because if we have $X$ a point $x \in D_1$ it is essientaly going to create a grid of sorts of points that cover $X$, creating a "mesh" of points where the openengs are at least 1 unit,



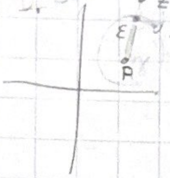




ii) Prove $D_1$ is a countable set.

↳ Assume $D_1$ is uncountable, so then $D_1$ must have a L.P, or some $p$ s.t $\forall \varepsilon > 0$, $(B(p;\varepsilon) - \{p\}) \cap D_1 \neq 0$.

However, if $1 > \varepsilon > 0$, the Ball around $p$ will not hit any other points in $D_1$. Thus, $p$ is not a L.P $\rightarrow\leftarrow$

Thus $D_1$ is Countable.




b) i) $D_2$ is a set of all points where each point is at least $\frac{1}{2}$ units away from eachother. Again, it is going to create a "mesh" of points whose openings are all $\frac{1}{2}$ unit, so a Unit $\frac{1}{2}$ mesh?

ii $D_1 \subseteq D_2$, because all points that are greater than 1 unit are greater than $\frac{1}{2}$ unit

iii) No, same as the argument before, there cannot exist a L.P, so therefore $D_2$ cannot be uncountable.

c) $D_1 \subseteq D_2 \subseteq D_3 \subseteq \cdots \subseteq D_n$, all of which are Countable sets.

d Since $D_1$ through $D_n$ are all countable, then $\bigcup_{n=1}^{\infty} D_n$ is also Countable.

e) $D = \bigcup_{n=1}^{\infty} D_n$

Prove $D$ is dense in $X$ → Any open set $U$ in $X$, $U \cap D \neq \emptyset$.

Let $U$ be an open set in $X$. Thus, $\forall x \in U$, there exists $B(x, \varepsilon)$ s.t. $\varepsilon > 0$ and $B(x, \varepsilon) \subseteq U$.

writing
This feels funky.

If $D$ is dense, $U \cap D \neq \emptyset$., so there $\exists y \in D$ s.t $y$ is also in $B(x, \varepsilon)$.

Thus $y \in \bigcup_{n}^{\infty} D_n$, so $y$ is in at least 1 set, say $y \in D_n$., so $d(x,y) < \varepsilon$.

↳ basically no matter How small $n$ is $\left(d(x,y) \geq \frac{1}{n}\right)$, we can always find an $N < n$, where the Distance between points in $D_N < \varepsilon$.

Thus, $\exists y \in B(x, \varepsilon)$ where $y \in D$, so $D$ is dense in $X$.