Homework #2 - Math 626

M[illegible] #Math 626/Module 1 [illegible]ehn

A1)

Late one night, a weary trans-dimensional traveler arrives at an extremely large and luxurious hotel. Calling it extremely large may not quite do it justice, since this hotel happens to have a countably infinite number of rooms (that is, it has an infinite number of rooms that have been labelled using the Natural Numbers: 1, 2, 3, ...).

Our exhausted (yet intrepid) traveler is disheartened when the night clerk informs him that the hotel has no vacancies – every room is occupied. Just then, the night manager arrives on the scene. "We can certainly accommodate you in our fine hotel." she exclaims. "To make room for you, we simply must contact each guest and give them very specific instructions about which new room they are being moved into". In order for the instructions to be specific enough, each person must be able to use the instructions to figure out the specific number of their new room.

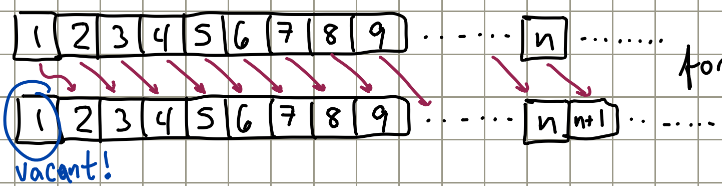
Countably infinite → 1-1 correspondance w/ $\mathbb{N}$

a) What set of instructions can the night manager use to reassign the guests in order to free up a room for our weary traveler?

The night manager could tell each person to go to the room number 1 greater than they are in now.

↳ each $n \rightarrow n+1$

- We know that there are an infinite number of $n \in \mathbb{N}$. We also know that for any value in $\mathbb{N}$, adding 1 will also give a Natural number. Because of this fact my brain immediately jumped to adding 1 to each room number, giving guests very specific instructions, while opening up room 1 for the traveler.



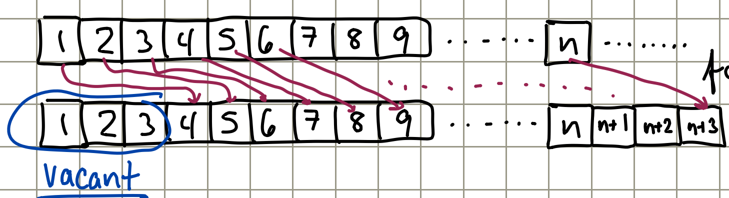
for any $n \in \mathbb{N}$, $n+1 \in \mathbb{N}$, so this must be a valid movement.

b) What if our traveler has two friends with him who also need rooms? Could you modify the instructions to create space for all three of them?

For this problem, using the same logic from the previous problem, if we can add 1 to any $n \in \mathbb{N}$, we can add any k value, in this case 3.

So, the night manager could tell each person to go to the room number twice as big than they are in now

↳ $n \rightarrow n+3$



for any $n \in \mathbb{N}$, $n+3 \in \mathbb{N}$, so this must be a valid movement.

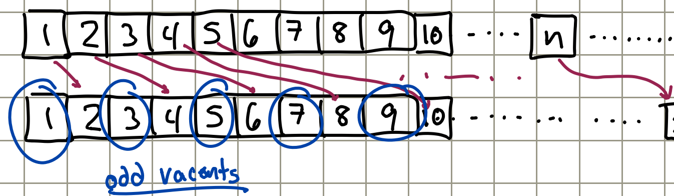
c) What if a countably infinite number of new guests arrive? Is it still possible to give instructions that would assign rooms for both the previous guests **and** this entire new collection of guests?

This one is tricky, because you can't tell each person to move $n+\infty$ # of rooms. Instead, I thought about what are countably infinite sets that exist inside of $\mathbb{N}$? The answer is odd num. and even num.

So, the night manager could tell each person to go to the room number that is twice as big than they are in now

↳ $n \to 2n$

This would take every guest currently in a room and move them to an even numbered room, leaving all of the odd room vacant for the new guests.



for any $n \in \mathbb{N}$, $2n \in \mathbb{N}$, so this must be a valid movement.

A2) A vendor at a farmer's market devised a clever method for weighing produce using an old-fashioned pan balance scale. In this system, the produce was placed in one pan, and then weights were placed into the other pan until the scale is balanced. The vendor has a ¼ pound weight, a ½ pound weight, and five other weights, each having a different positive integer value. The vendor claims that it is possible to weigh any purchase from 0 to 31 pounds (to within an accuracy of ¼ of a pound) using this system.

a) Find the weight of each of the five positive integer valued weights used by this vendor.

For this problem, I started by writing down all values. Since we want to be accurate to ¼ of a pound, in order to be able to tell if something is 1^{lb}, we need to have ¾ and $1\frac{1}{4}$ pounds.

0	Obvious	15	8+4+2+1
$\frac{1}{4}$	¼	16	* 16
$\frac{1}{2}$	½	17	16+1
1	* 1	18	.
2	* 2	19	.
3	1+2	20	.
4	* 4	21	
5	4+2	22	.
6	4+1	23	.
7	4+2+1	24	.
8	* 8	25	.
9	8+1	26	.
10	8+2	27	.
11	8+2+1	28	.
12	8+4	29	.
13	8+4+1	30	
14	8+4+2	31	16+8+4+2+1 ✓

* To get to 1 pound, ½+¼ only gets to ¾, so we add a 1^{lb} weight
* Using all wts. 1+½+¼ = 1¾, so we add a 2^{lb} wt.
* Using all current wts, ¼+½+1+2 = 3¾, so we add a 4^{lb} wt.
* Using all current wts, they only add up to 7¾, so add a 8^{lb} wt.
  ↳ It's easy to see the pattern from here.
* Using all wts, the total is 15¾, so we add a 16^{lb} wt.

↳ If we added these 5 wts together, (¼+½+1+2+4+8+16) we get 31¾, so we can weigh up to 31^{lbs}

- We can also figure out wts. between the integers. For example: 18¾ can be measured w/ 16+2+¼+½

- So the 5 integer wts that we add are:
  1, 2, 4, 8, 16 (I noticed it made the pattern of 2^n, where $n=0,1,2,3,4$)

- We know that any whole number can be expressed as a sum of some 2^n, so any value up to the max amount can be expressed.

b) How many additional weights would be needed to weigh anything up to *at least* 100 pounds (to the same level of accuracy)?

Current wts.: $\frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16$

↳ Continuing from where we left off on the previous part, all wts added up to $31\frac{3}{4}$ lbs. That means to get 32 lbs, we add a weight of that size.

* $\frac{1}{4}+\frac{1}{2}+1+2+4+8+16+32 = 63\frac{3}{4}$

* Since the max after that wt is $63\frac{3}{4}$, To reach 64, we add that wt.

This means that the new max wt. that we can measure is: $\frac{1}{4}+\frac{1}{2}+1+2+4+8+16+32+64 = 127\frac{3}{4}$

Since $121\frac{3}{4} > 100$, we only need 2 additional weights.

Prove we can weigh 100 lbs:

$100 = 64+32+4$

A3)

Suppose 5 people write their names on separate pieces of paper and put them into a hat. After mixing them up, each person randomly draws one name out of the hat (without replacement).

a) What is the probability that no one gets their own name?

Since we have 5 people, We know there are $5! = 120$ ways to distribute names. I started by Simplifying the problem to 4 people (24 combinations) → I tried 3, but didn't see a clear pattern, or method.

A B C D

~~A B C D~~ ~~B A C D~~ ~~C A B D~~ D A B C
~~A B D C~~ B A D C C A D B ~~D A C B~~
~~A C B D~~ ~~B C A D~~ ~~C B A D~~ ~~D B A C~~
~~A C D B~~ B C D A ~~C B D A~~ ~~D B C A~~
~~A D B C~~ B D A C C D A B D C A B
~~A D C B~~ ~~B D C A~~ C D B A D C B A

$24-(8+6+1)$
$24-15 = 9$ ✓

In thinking of a systematic way to eliminate options, I started by eliminating all where all 4 got their own name 1 1

• Then, all where 3 people got their own name 1 - already counted

Then, 2 people got own name. 7 → 1 already counted 6

Then, 1 person got their own name. 15 → 7 already counted 8

↳ This reminded me of doing derangements in Combinatorics, so I approached it that way. *

Because you have to account for all variations of getting names:

$$|\text{None}| = \underbrace{|\text{All}|}_{\text{Ways to}} - \Big(\underbrace{|1\text{ name}|}_{\text{Ways to}} - \underbrace{|2\text{ names}|}_{\text{Overcounted}} + \underbrace{|3\text{ names}|}_{\text{Removed 3 names}} - \underbrace{|4\text{ name}|}_{\text{Over counted}} + \underbrace{|5\text{ names}|}_{\text{Add back in}}\Big)$$

So, for when $n=5$:

• $5!$ ways to distribute

(+) • 1 person gets own name: $\underbrace{\binom{5}{1}}_{\text{Pick who gets own name}} \underbrace{(5-1)!}_{\text{ways to distribute the rest of names}} = 5\cdot 4! = 5!$

(−) • 2 people get own name: $\binom{5}{2}(5-2)! = \frac{5!}{2!3!}\cdot 3! = \frac{5!}{2!}$

(+) • 3 people get own name: $\binom{5}{3}(5-3)! = \frac{5!}{3!2!}2! = \frac{5!}{3!}$

(−) • 4 people get own name: $\binom{5}{4}(5-4)! = \frac{5!}{4!1!}\cdot 1! = \frac{5!}{4!}$

(+) • 5 people get own name: $\binom{5}{5}(5-5)! = \frac{5!}{5!0!}0! = \frac{5!}{5!}$

So, the number of options where no one drew their own name is:

$$= 5! - \left(5! - \frac{5!}{2!} + \frac{5!}{3!} - \frac{5!}{4!} + \frac{5!}{5!}\right)$$
$$= 5!\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)$$
$$= 120\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right)$$
$$= 120\left(\frac{11}{30}\right)$$
$$= 44$$

So the probability that no one gets their own name is

$$P(\text{no name}) = \frac{44}{5!} = \frac{44}{120} \approx 0.3667 \approx 36.67\%$$

b) What is the probability that exactly one person gets their own name?

For this one, I approached it the same way as the previous problems with derangements.

- First, again we know that there are 5! ways to arrange names.
- Then to first choose which student picks their own name, we have $\binom{5}{1} = \frac{5!}{1!4!} = 5$ ways to do that.
  - ↳ Then, for the remaining 4 students, we have to eliminate these possibilities
    - 1 of remaining 4 gets own name: $\binom{4}{1}(4-1)! = \frac{4!}{1!3!}3! = 4!$
    - 2 of remaining 4 gets own name: $\binom{4}{2}(4-2)! = \frac{4!}{2!2!}2! = \frac{4!}{2!}$
    - 3 of remaining 4 gets own name: $\binom{4}{3}(4-3)! = \frac{4!}{3!1!}1! = \frac{4!}{3!}$
    - 4 of remaining 4 gets own name: $\binom{4}{4}(4-4)! = \frac{4!}{4!0!}0! = \frac{4!}{4!}$

So, the number of options where no one drew their own name is:

$$= \underbrace{5}_{\text{Who gets their name}}\underbrace{\left(4! - \left(4! - \frac{4!}{2!} + \frac{4!}{3!} - \frac{4!}{4!}\right)\right)}_{\text{No one else can draw own name}}$$
$$= 5\left(4!\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)\right)$$
$$= 5\left(4!\left(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right)\right)$$
$$= 5\left(24\left(\frac{3}{8}\right)\right)$$
$$= 5(9)$$
$$= 45$$

So the probability that exatly one gets their own name is

$$P(\text{one name}) = \frac{45}{5!} = \frac{45}{120} = 0.375 = 37.5\%$$

I knew that I could use derangements for this problem because derangements are just the number of elements where no element appears its original position.

B3. What happens the value of the probabilities you found in A3 as the value of n increases (up from 5 to greater positive integers)?

If we use the formula for derangements (Which is just the number of permutations where no element is in its original position):

We know that to find the number of derangements, the formula is $\underbrace{D(n) = n!\sum_{j=0}^{n}\frac{(-1)^j}{j!}}_{\text{formula from Math 511 :)}}$. We want to know what happens as $n \to \infty$.

However, because we are looking at Probability, we should consider $\frac{D(n)}{n!}$ ← Total arrangements

n! Disappeared when we divided

$$\frac{D(n)}{n!} = \sum_{j=0}^{n}\frac{(-1)^j}{j!}$$

$$\lim_{n\to\infty}\frac{D(n)}{n!} = \underbrace{\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}}$$

like Taylor Series for $e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}$, so let $x = -1$

$$\lim_{n\to\infty}\frac{D(n)}{n!} = e^{-1}$$

$$\lim_{n\to\infty}\frac{D(n)}{n!} = \frac{1}{e} \approx 0.3679 = 36.79\%$$

So, as n gets bigger, the probability approches $\frac{1}{e}$

I verified this result for afew $n > 5$:

$n = 6$: $\frac{6!\sum_{j=0}^{6}\frac{(-1)^j}{j!}}{6!} = 0.36805 \approx 36.81\%$

$n = 10$: Using Maple $= 0.3679 = 36.79\%$

$n = 100$: Using Maple $= 0.3679 = 36.79\%$

A4)

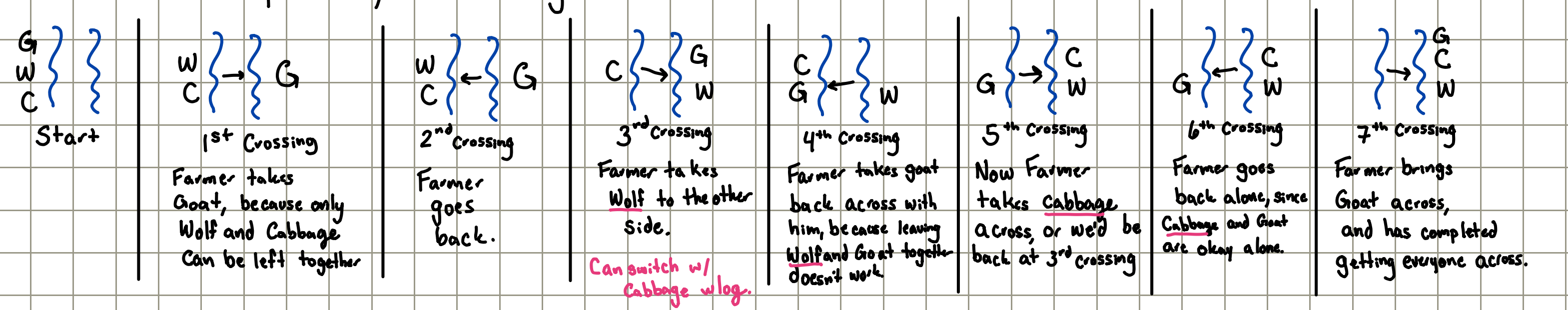
Consider the following scenario:

Once upon a time, a farmer went to a market and purchased a goat, and wolf, and a bag of cabbage (I am not sure why he wanted a wolf). On his way home, the farmer came to the bank of a river. Since the bridge had been destroyed in a flood the previous spring, he had to cross it by boat. Fortunately, he had left his trusty rowboat on the near bank. Unfortunately, the boat was quite small, so in crossing the river by boat, the farmer could carry only himself and a single one of his items: the goat, the wolf, or the bag of cabbage.

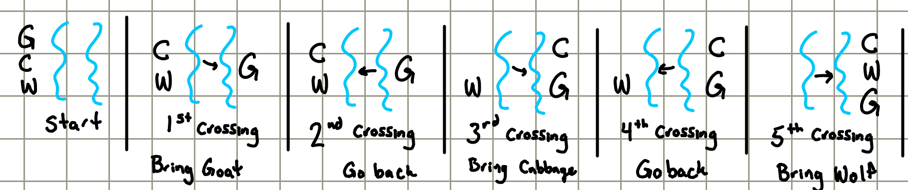
- If left unattended together, the wolf would eat the goat
- Similarly, if left unattended, and the goat would eat the cabbage.

Help the farmer find a way to transport himself and all of his purchases to the far bank of the river in a way that will keep each item intact. You should find a specific method that the farmer can use, or demonstrate that no effective strategy exists. If a strategy does exist, try to find the most efficient method possible.

For this problem, I drew diagrams:



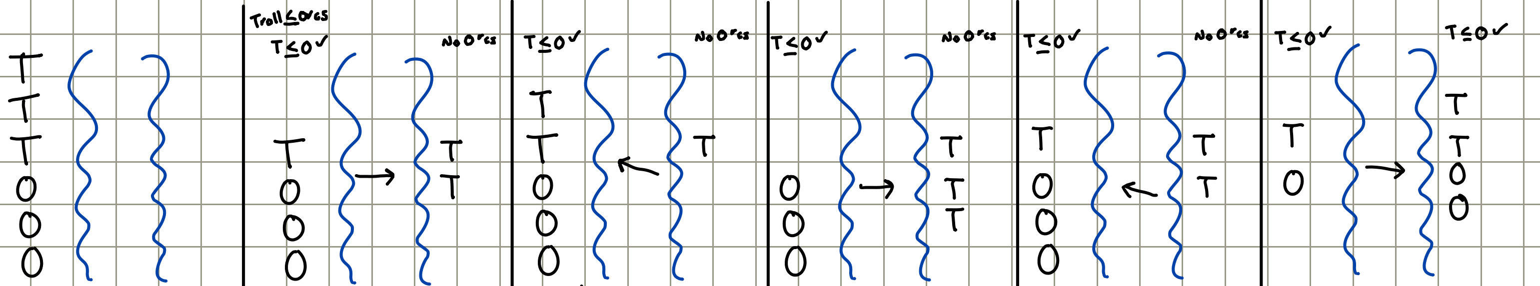
When I was constructing this method, I used a lot of trial and error, and at each step, making sure I didn't violate one of the conditions. To make sure I had the most efficient solution, I looked at what the number of crossings would be with no conditions.



We could do this in 5 crossings, but because we have to bring the goat across and back twice to prevent eating, we add 2 crossings for a total of 7, showing its the most efficent.

- We know it must be an odd # of crossings so we end up on the right bank. (6 would end on left)

B4) Suppose that three trolls and three orcs are travelling together and arrive at the same river with a similar boat. As before, a maximum of two of them can fit into the boat at the same time. The goal is to get them all safely across the river. Complicating matters is the fact that trolls are bigger than orcs and are always hungry, so if the number of trolls ever exceeds the number of orcs, the trolls will eat the orcs. Find a specific method that can be used to safely transport all six of them across the river or demonstrate that no effective strategy exists. If a strategy does exist, try to find the most efficient method.



Start

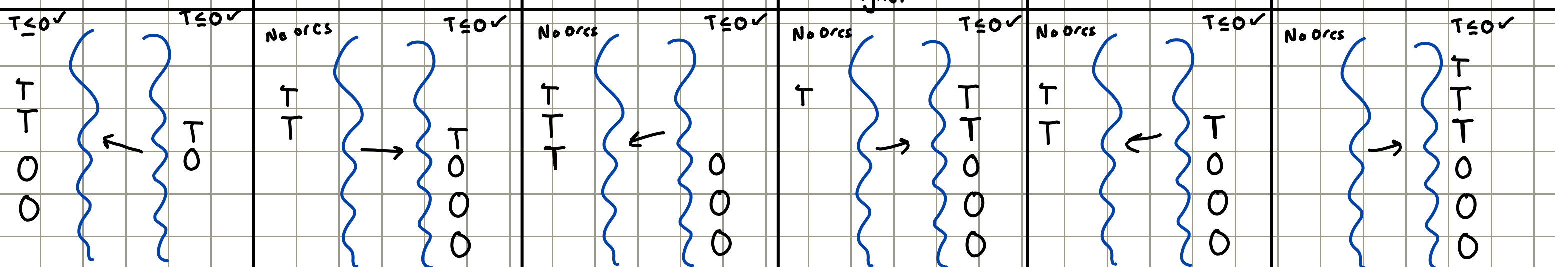
1st Crossing
First Two trolls go across. On the left bank Trolls ≤ Orcs, so noone is eaten.

2nd Crossing
1 of the trolls rows back across.

3rd Crossing
Both trolls row across the river
- 2 orcs, orc on left is eaten
- orc and Troll, orc is eaten on right.

4th Crossing
1 of the trolls Rows back.

5th Crossing
2 orcs go across.
- If Troll and orc, too many trolls on right and orc eaten



6th Crossing
It must be a troll and orc rowing back across.
- 2 orcs, back to step 2
- 2 Trolls, orc on left is eaten.

7th Crossing
2 orcs back across
- If 2 Trolls, orc on the right gets eaten
- If 1 of each, back to Step 5.

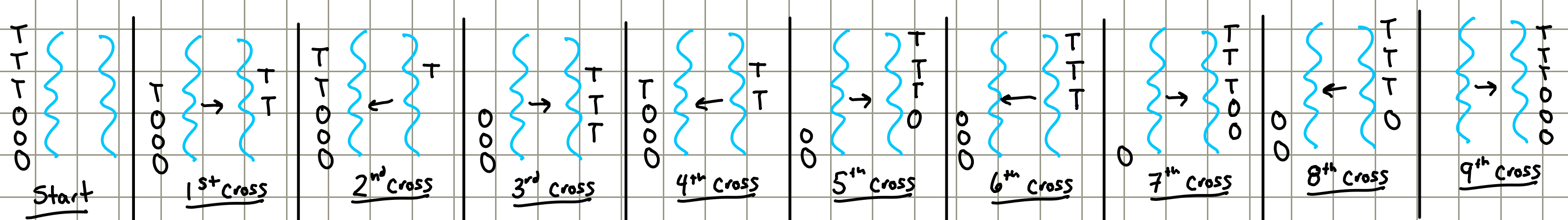
8th Crossing
Troll back across
- If orc, eaten on right
- If orc + Troll, orc eaten
- If 2 orcs, back to Step 6.

9th Crossing
2 trolls across
- 1 troll, back to Step 7.

10th Crossing
1 troll back across.
- 2 trolls → back to step 8
- 2 orcs → Creates extra Steps
- 1 orc would also be okay

11th Crossing
2 Trolls across and Success!

When I was constructing this method, I used a lot of trial and error, and at each step, looked a what other combos would do. Like part A, to make sure I had the most efficient solution, I looked at what the number of crossings would be with no conditions.



We could do this in 9 crossings, (2 over → 1 back for the first 4 round trips, and 2 over for the 9th cross). However, there is no way
Net 4 on right
to not violate the condition that # Trolls ≤ # orcs. So, we add an extra there and back moving an orc we already moved back so # Trolls ≤ # Orcs.

- We know it must be an odd # of crossings so we end up on the right bank. (10 would end on left)