Homework #2

1. a) Reflexive: Let $(a,b) \in R$. Because $a+b = a+b$, we know that R is reflexive for all $(a,b) \in A$.

Symmetry: Let $(a,b), (c,d) \in A$. Assume $(a,b)R(c,d)$. Thus, $a+b = c+d$. Thus, it follows that $c+d = a+b$, showing that $((c,d),(a,b)) \in R$, and its symmetric

Transitive: Let $(a,b), (c,d), (e,f) \in A$. Suppose $((a,b),(c,d))$ and $((c,d),(e,f)) \in R$. Thus, $a+b = c+d$ and $c+d = e+f$.

Taking $a+b = c+d$, we can substitue, giving us $a+b = e+f$, Proving $((a,b),(e,f)) \in R$.

Since R is reflexive, symmetric & transitive, it must be an equivalence relation. ■ :)

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b) $E_{(1,2)} = \{(c,d) \in \mathbb{R} \times \mathbb{R} \mid c+d = 3\}$

$1+2 = c+d$
$3 = c+d$

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c) $(1,2)R(x,y)$: $1+2 = x+y$
$3 = x+y$
$3-x = y$
$y = -x+3 \rightarrow$

$(3,4)R(x,y)$: $3+4 = x+y$
$7 = x+y$
$7-x = y$
$y = -x+7$

These are both lines with a slope of -1.
So, every equivalence class can be shown on a graph as $y = -x+k$, where $k = a+b$, where $(a,b) \in R$.

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2. a) $\text{dom} f = [0, \pi]$
$\text{rng} f = [0, 1]$

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b) f is <u>not</u> surjective. In order to be surjective, when $f: A \rightarrow B$, $B = \text{rng} f$.

Since $\text{rng} f = [0,1]$, but $\text{cdom} f = \mathbb{R}$, and $[0,1] \neq \mathbb{R}$, it is not surjective.

f is <u>not</u> injective. To be injective, if $f(a) = f(a')$, then $a = a'$.
However, consider $a = 0$ and $a' = \pi$.

$f(0) = f(\pi) \rightarrow \sin(0) = \sin(\pi) \rightarrow 0 = 0$. but, $a \neq a'$, so not injective!

2 c i) $f([0,\frac{\pi}{2}]) = [0,1]$

ii) $f^{-1}([0,\frac{1}{2}]) = [0,\frac{\pi}{6}] \cup [\frac{5\pi}{6},\pi]$

iii) $f(f^{-1}([\frac{\sqrt{3}}{2},2])) = [\frac{\sqrt{3}}{2},1]$

$\hookrightarrow f^{-1}([\frac{\sqrt{3}}{2},2]) = [\frac{\pi}{3},\frac{2\pi}{3}]$

$\hookrightarrow f([\frac{\pi}{3},\frac{2\pi}{3}]) = [\frac{\sqrt{3}}{2},1]$

iv $f^{-1}([-1,0)) = \emptyset$

Empty set!

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d) $C_1 = [0,\frac{\pi}{2}]$
$C_2 = [\frac{\pi}{2},\pi]$

$f(C_1 \cap C_2) = f([\frac{\pi}{2}]) = 1$

$f(C_1) = [0,1]$
$f(C_2) = [0,1]$ $\rightsquigarrow f(C_1) \cap f(C_2) = [0,1]$

$1 \neq [0,1]$

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3. $f: \mathbb{Z} \to \mathbb{Z}$

a) $f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x-1 & \text{if } x > 0 \end{cases}$

This is surjective, because every $x \in \mathbb{Z}$ has a solution. However, it is not injective because of 0 and 1.

$f(0) = 0$
$f(1) = 1-1=0$

$f(0) = f(1)$, but $0 \neq 1$.

b) $f(x) = 2x$.

This is injective, See below!

$f(x) = f(y)$
$2x = 2y$
$x = y$

This is not surjective, because you will only get even integers, never odd.

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4. $\mathbb{N} = \{1, 2, 3, \ldots\}$ $E = \{2, 4, 6, \ldots\}$

$f : \mathbb{N} \to E$

$f(x) = 2x$

To Prove a bijection, f must be both injective and surjective.

Let $x, y \in \mathbb{N}$ s.t. $f(x) = f(y)$. Thus, $2x = 2y$, and $x = y$, which means that f is injective.

Now, we need to show that f is surjective. To prove surjective, there must be an $x \in \mathbb{N}$ for every $e \in E$.

Let $e \in E$.

$$f(x) = e$$
$$2x = e$$
$$x = \frac{e}{2}$$

For every $e \in E$, there is an $x \in \mathbb{N}$, where $x = \frac{e}{2}$, so each e has a pre-image.

Since f is both injective and surjective, it is a bijection ■ :)

3.

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5. $f : A \to B$ Relation, R on A iff $f(x) = f(y)$
xRy

a) Reflexive: Let $x \in A$. Because $f(x) = f(x)$ it follows that xRx, for all x.

Symmetric: Let $x, y \in A$. Assume xRy, so $f(x) = f(y)$. Since $f(y) = f(x)$, we know that yRx, showing symmetry.

Transitive: Let $x, y, z \in A$. Assume xRy and yRz. Thus it follows that $f(x) = f(y)$ and $f(y) = f(z)$. Further, we know that $f(x) = f(y) = f(z)$, so $f(x) = f(z)$. This proves that xRz

Thus, R is an equivalence relation due to R being Reflexive, Symmetric, and transitive ■ :)

5b) $\forall x \in A$, $E_x = \{y \mid yRx\}$, $E = \{E_x \mid x \in A\}$ $g: A \to E$ by $g(x) = E_x$

To prove g is surjective, for every $E_y \in E$, there must be $x \in A$ s.t. $g(x) = E_y$.

Consider when $x = y$.

Let $E_y \in E$. So, $g(x) = E_y$

$g(y) = E_y$.

So, for every $E_y \in E$, there is an $x \in A$, when $x = y$ s.t. $g(x) = E_y$.

Proving surjective. ■ :)

* Might have done some math magic here.... not sure this is legal..

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c) $h: E \to B$ by $h(E_x) = f(x)$

Let $E_x, E_y \in E$ s.t. $h(E_x) = h(E_y)$

Thus, $h(E_x) = h(E_y)$

so $f(x) = f(y)$.

which means that xRy.

Math Magic??

Since R is an equivalence relation, yRx.

This means since xRy, $y \in E_x$ and because yRx, $x \in E_y$.

So, $E_x = E_y$, proving injection ■ :)

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d) $h \circ g: A \to B$ $h \circ g = f$

We need to show that $h \circ g(x) = f(x)$ $\forall x \in A$.

Let $x \in A$. $h \circ g(x) = h(g(x))$. by part b, we know that $g(x) = E_x$,

$h \circ g(x) = h(E_x)$, which we know by part c $h(E_x) = f(x)$,

$h \circ g(x) = f(x)$, proving that $h \circ g = f$ ■ :)