Homework #2 - Math 511

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1) Let $\mathcal{E}$ be a relation on $\mathbb{R} \times \mathbb{R}$, where $((a,b),(c,d)) \in \mathcal{E}$ if $\sqrt{a^2+b^2} = \sqrt{c^2+d^2}$.

(a) Prove that $\mathcal{E}$ is an equivalence relation on $\mathbb{R} \times \mathbb{R}$.

Pf: Reflexive:

Suppose $a, b \in \mathbb{R}$. Then, $\sqrt{a^2+b^2} = \sqrt{a^2+b^2}$. This is clearly true, so $\mathcal{E}$ is reflexive.

Symmetric: (show if $((a,b),(c,d)) \in \mathcal{E}$ then $((c,d),(a,b)) \in \mathcal{E}$

Let $a,b,c,d \in \mathbb{R}$ and $\sqrt{a^2+b^2} = \sqrt{c^2+d^2}$. By properties of equality, $\sqrt{c^2+d^2} = \sqrt{a^2+b^2}$. So, $((c,d),(a,b)) \in \mathcal{E}$ and it is symmetric.

Transitive:

Let $a,b,c,d,e,f \in \mathbb{R}$ and $\sqrt{a^2+b^2} = \sqrt{c^2+d^2}$ and $\sqrt{c^2+d^2} = \sqrt{e^2+f^2}$. Using substitution, we know that $\sqrt{a^2+b^2} = \sqrt{e^2+f^2}$ and so the relation is transitive.

Thus, $\mathcal{E}$ is an equivalence relation.

(b) Describe geometrically, the equivalence class of the point $(0,1)$.

The equivalence class of the point $(0,1)$ is all points (a,b) s.t $a,b \in \mathbb{R}$ and $\sqrt{0^2+1^2} = \sqrt{a^2+b^2}$

So, $\sqrt{0^2+1^2} = \sqrt{a^2+b^2}$

$1 = \sqrt{a^2+b^2}$

$1 = a^2+b^2$

Geometrically, this would be all points on a circle of radius 1, centered at the origin.

Written in the set form, it would be:

$\mathcal{E}((0,1)) = \{a,b \in \mathbb{R} \mid a^2+b^2 = 1\}$

(c) Give a general description for what the equivalence classes look like.

Like in part b, we know that equivalence classes will generate points on a circle of radius $\sqrt{a^2+b^2}$ centered around the origin.

Written in set form, $\mathcal{E}((a,b)) = \{c,d \in \mathbb{R} \mid \sqrt{a^2+b^2} = \sqrt{c^2+d^2}\}$, or any other points that are the same distance from the origin as (a,b).

(d) There is exactly one equivalence class with finitely many elements. What is it? Does it look like the other equivalence classes?

The only equivalence class with a finite amount of elements is $\mathcal{E}((0,0))$.

This is because $\mathcal{E}((0,0))$ will only contain the point $(0,0)$. Every other equivalence class will contain an infinite amount of points, and generate a circle, where $\mathcal{E}((0,0))$ will just be a single point, not a circle.

2) Pg 40 #10 How many ways to split a group of 10 people into 2 groups of size 3 and one group of size 4?

We know that there are 10! permutations of 10 people, so $|A| = 10!$

For the size of the equivalence classes, we have a few things to consider:
↳ • 3! ways to order the first group of 3
• 3! ways to order the second group of 3
• 4! ways to order the group of 4
• The two groups of 3 can swap and still be equivalent (2)

So $C = 3!3!4!\,2$

Using the Equivalence principle, there are $\frac{10!}{3!3!4!2} = \frac{3628800}{6\cdot 6\cdot 24\cdot 2} = \frac{3628800}{1728} = 2100$ ways to split the group

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3) Pg 40 #14 Prove the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ using the equivalence principle

We know that there are $n!$ distinct permutations of $[n]$, so $|A| = n!$
Using the equivalence principle, $k = \frac{|A|}{C}$, we can rewrite that as $|A| = kC$.
Because k is the number of equivalence classes, we know that $k = \binom{n}{k}$, as we are choosing k-elements from n. Additionally, C is the size of our equivalence classes. Thus, $C = k!(n-k)!$
This is due to the fact that we first arrange our k elements in $k!$ ways, and then the remaining elements, $n-k$, can be arrange in $(n-k)!$ ways.

So $|A| = kC$
$n! = \binom{n}{k} k!(n-k)!$ ↗ Solving for $\binom{n}{k}$ gives $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ ∎

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4) Pg 47 #3 Let $n \geq 1$, and let S be an $(n+1)$-subset of $[2n]$. Prove that there exists two numbers in S whose sum is $2n+1$.

Consider the following partition of $[2n]$: $\{\{1, 2n\}, \{2, 2n-1\}, \{3, 2n-2\}, \ldots, \{n, n+1\}\}$
Each pair in the partition adds to give $2n+1$, and there are n-elements in the partition.
In general, each pair in the partition can be written as $\{k, 2n+1-k\}$, where $1 \leq k \leq n$.

Now, we know $S \subseteq [2n]$, and assume S contains one element from each pair in the partition. However, because $|S| = n+1$, this means that there must be at least one pair $\{k, 2n+1-k\}$, where both $k, 2n+1-k \in S$. Because the sum of the pair is $2n+1$, there exists 2 numbers in S whose sum is $2n+1$ by the pigeonhole principle.

5) Page 47 #6ab Consider any 5 points in the plane that have integer coordinates

a) Prove that there are two points s.t. the midpoint of the line segment joining those two points also has integer coordinates.

Pf: Consider the points $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5)$ where $x_i, y_i \in \mathbb{Z}\ \forall i$.

We know that midpoint between 2 points, (x_i, y_i) & (x_j, y_j) is found by the formula:

$$M = \left(\frac{x_i + x_j}{2}, \frac{y_i + y_j}{2}\right)$$

For M to have integer coordinates, $x_i + x_j$ must be even and $y_i + y_j$ must be even. This can only happen if x_i and x_j are both even or both odd, and the same for y_i & y_j.

Each (x_i, y_i) can be classified as: (even, even), (even, odd), (odd, even), or (odd, odd). For both $x_i + x_j$ and $y_i + y_j$ to add to give an even integer, both (x_i, y_i) and (x_j, y_j) must be the same pariety. Since there are five points and only 4 pariety classificatrons, by the pigeon hole principle, at least two points must have the same pariety classification, and that midpoint will have integer coordinates. ∎.

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b) Show that this is not necessarily true with only four points.

Pf: Consider the points $A = (0,0)$, $B = (0,1)$, $C = (1,0)$, and $D = (1,1)$.

This creates the line segments $\overline{AB}, \overline{AC}, \overline{AD}, \overline{BC}, \overline{BD}, \overline{CD}$. Now, we find the midpoint of each segment.

Midpoint of $\overline{AB}$: $\left(\frac{0+0}{2}, \frac{0+1}{2}\right) = \left(0, \frac{1}{2}\right)$ which doesn't have only integer coordinates

Midpoint of $\overline{AC}$: $\left(\frac{0+1}{2}, \frac{0+0}{2}\right) = \left(\frac{1}{2}, 0\right)$ which doesn't have only integer coordinates

Midpoint of $\overline{AD}$: $\left(\frac{0+1}{2}, \frac{0+1}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$ which doesn't have only integer coordinates

Midpoint of $\overline{BC}$: $\left(\frac{0+1}{2}, \frac{1+0}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$ which doesn't have only integer coordinates

Midpoint of $\overline{BD}$: $\left(\frac{0+1}{2}, \frac{1+1}{2}\right) = \left(\frac{1}{2}, 1\right)$ which doesn't have only integer coordinates

Midpoint of $\overline{CD}$: $\left(\frac{1+1}{2}, \frac{0+1}{2}\right) = \left(1, \frac{1}{2}\right)$ which doesn't have only integer coordinates

6) Pg 47 #7 Prove that the result of the Erdős-Szekeres thrm is best possible, in that it is possible for a sequence of n^2 distinct real #'s to have neither an increasing subsequence of length $n+1$ nor a decreasing subsequence of length $n+1$.

Example: $n=3$

x	3	2	1	6	5	4	9	8	7
$LIS(x)$	3	3	3	2	2	2	1	1	1
	n	$n-1$	$n-2$	$2n$	$2n-1$	$2n-2$	n^2	n^2-1	

$n: 4 \quad | 4 \; 3 \; 2 \; 1 \,|\, 8 \; 7 \; 6 \; 5 \,|\, 12 \; 11 \; 10 \; 9 \,|\, 16 \; 15 \; 14 \; 13 \,|$

$n: \quad n \; n-1 \; n-2, \cdots 1 \,|\, 2n \; 2n-1 \cdots n+1 \,|\, 3n \cdots 2n+1 \,|\, n^2 \cdots\cdots n^2-(n-1)$

Pf Let n be a positive integer. We can instruct a sequence of n^2 distinct real #'s which have neither an increasing subsequence of length $n+1$ nor a decreasing subsequence of length $n+1$

Consider the integers $\{1, 2, \cdots, n^2\}$. We will divide the integers into n blocks, each containg n elements.

For the k^{th} block, where $1 \leq k \leq n$, block will consist of the integers $kn, kn-1, \cdots, kn-n+1$

So, the whole sequence is:

$$n, n-1, n-2, \ldots, 1, 2n, 2n-1, \ldots, n+1, \cdots, n^2, n^2-1, \cdots, n^2-n+1$$

No Increasing Subsequence of length $n+1$:

Because each block is arranged in decreasing order, each subsequence can only contain 1 element from each block, for the longest sub-sequence of length n.

No Decreasing Subsequence of length $n+1$:

Within each block, elements are decreasing. Since elements of all subsequent blocks are larger than the last element of the current block, to get a subsequence of $n+1$ length, you would have to have elements from multiple blocks, which is not possible.

So, we have proved that Erdős-Szekeres is best possible. ∎

7) Let $x_1, x_2, \ldots, x_n$ be a sequence of integers. Show that there exist integers i and j with $1 \le i \le j \le n$ such that the sum $\sum_{t=i}^{j} x_t$ is divisible by n. (Hint: Think about remainders when dividing by n.)

We know that if something is divided by n, there are n possible remainders $\{0,1,2,\ldots,n-1\}$

Let $S_k = x_1 + x_2 + \ldots + x_n$ for the 1st k elements in the list.
This gives $n+1$ S_ks

example) 1st 0 elements $S_0 = 0$
1st 3 elements $S_3 = x_1 + x_2 + x_3$
1st n elements $S_n = x_1 + x_2 + \ldots + x_n$

By the pigeohole prinicple, this means at least two S_ks will share a remainder. We know that by modular arithemetic, that if $a \% n = b \% n$, then $a - b \% n = 0$, or n will evenly divide the difference.

So, if $S_j \% n = k$ and $S_{i-1} \% n = k$, then $S_j - S_{i-1} \% n = 0$ and we have found elements i and j, and that $\sum_{t=i} x_t$ will divide by n.

We use % for mod in Computer Science, I can't remember what you used, and couldn't find in notes.

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8) Pg 57 #4 Consider the possible functions $f: [7] \to [9]$

We know there are 9^7 possible functions.

a) How many have $f(3) = 8$?

One of the 7 inputs is fixed, but the other 6 have 9 options, so the number of functions with $f(3) = 8$ is $9^6 = 531{,}441$

How many have $f(3) \neq 8$?

Since there are 9^7 possible functions, and we don't want any of the possibities where $f(3) = 8$, there ae $9^7 - 9^6 = 4{,}251{,}528$ functions where $f(3) \neq 8$.

b) How many have $f(1) \neq 5$ and are one-to-one?

There are 8 choices for what $f(1)$ could be.
One $f(1)$ is chosen, there are 6 remaining inputs and 8 remaining outputs. Since it must be one-to-one, this can be expressed as $(8)_6$.

So there are $8 \cdot (8)_6 = 161{,}280$ functions.

c) How many have $f(i)$ even, for all i?

There are four even numbers in $[9]$: $\{2,4,6,8\}$
Since each of the seven inputs must go to one of those four outputs, there are $4^7 = 16{,}384$ functions

d) How many have $rng(f) = \{5, 6\}$?

Since there are two outputs, we begin with 2^7 functions. However, since we must have both $5 + 6$ in the $rng(f)$, we do not want to include the function where every $f(i) = 5, \forall i \in [7]$ and where every $f(i) = 6, \forall i \in [7]$.

So, there are $2^7 - 2 = 126$ functions

e) How many in which f^{-1} is not a function?

If a function is one to one, that would mean that f^{-1} is a function. There are $(9)_7$ functions where f is one-to-one.

Since I want all non-injective functions, there are $9^7 - (9)_7 = 4{,}601{,}529$

9) Pg 57 # 10 For $n \geq 1 + k \geq 1$, $(n)_k = n \cdot (n-1)_{k-1}$

Let S be the set of ways to distribute k distinct concert tickets to n distinct friends.
So $|S| = (n)_k$

Let A be one of the tickets. There are n possible friends that ticket could go to. After assigning that ticket, there are now $n-1$ friends and $k-1$ tickets, so $(n-1)_{k-1}$ ways to distribute. So, $|S| = n \cdot (n-1)_{k-1}$

Thus, we have proven that $(n)_k = n \cdot (n-1)_{k-1}$ ■

10) Give a combinatorial proof that $\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}$.

Let S be the total set of ways to form a committee of k-people from n-people + select a leader, of any size k.
This can be done by $k \cdot \binom{n}{k}$ ways. We are looking for the total # of ways to get a committee of any size, so we sum all possible committee sizes together,

So $|S| = \sum_{k=0}^{n} k \cdot \binom{n}{k}$

Begin by picking a leader from n-people. Then, we can create subsets from the remaing $n-1$ people to form all possible committe sizes.

So $|S| = n \cdot 2^{n-1}$

Thus, we have shown that $\sum_{k=0}^{n} k \binom{n}{k} = n \cdot 2^{n-1}$ ■

11) Page 65 #4c

Prove $\binom{0}{m}+\binom{1}{m}+\binom{2}{m}+\cdots+\binom{n}{m}=\binom{n+1}{m+1}$

Pf: Suppose we want to select a committee of $m+1$ people from a group of $n+1$ people. There are $\binom{n+1}{m+1}$ ways to do that.

Condition on who is the leader in the commitee.

Leader is person 1: Rest of comitte is formed by $\binom{0}{m}$

Leader is person 2: Rest of comitte is formed by $\binom{1}{m}$

$\vdots$

Leader is person $n+1$: Rest of comitte is formed by $\binom{n}{m}$

Adding the all together gives all ways to form the committee.

So, $\binom{0}{m}+\binom{1}{m}+\binom{2}{m}+\cdots+\binom{n}{m}$

Thus, $\binom{0}{m}+\binom{1}{m}+\binom{2}{m}+\cdots+\binom{n}{m}=\binom{n+1}{m+1}$ $\blacksquare$.

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12) Page 65 #47

Prove $\left(\!\binom{1}{k-1}\!\right)+\left(\!\binom{2}{k-1}\!\right)+\left(\!\binom{3}{k-1}\!\right)+\cdots+\left(\!\binom{n}{k-1}\!\right)=\left(\!\binom{n}{k}\!\right)$

Pf: Suppose we want to choose k-doughnuts from n varieties. There are $\left(\!\binom{n}{k}\!\right)$ ways to do that.

Condition on the location of the last doughnut:

Last doughnut in variety 1: $\left(\!\binom{1}{k-1}\!\right)$ to choose remaining doughnuts

Last doughnut in variety 2: $\left(\!\binom{2}{k-1}\!\right)$

Last doughnut in variety 3: $\left(\!\binom{3}{k-1}\!\right)$

So, $|S|=\left(\!\binom{1}{k-1}\!\right)+\left(\!\binom{2}{k-1}\!\right)+\cdots+\left(\!\binom{n}{k-1}\!\right)$

$\vdots$

Last doughnut in variety n: $\left(\!\binom{n}{k-1}\!\right)$

So, $\left(\!\binom{1}{k-1}\!\right)+\left(\!\binom{2}{k-1}\!\right)+\left(\!\binom{3}{k-1}\!\right)+\cdots+\left(\!\binom{n}{k-1}\!\right)=\left(\!\binom{n}{k}\!\right)$ $\blacksquare$

13) Page 65 #7f

Determine the # of solutions to each of the following equation:

$$z_1 + z_2 + z_3 + \frac{1}{2} z_4 = \frac{11}{2}$$

Begin by Clearing fractions:

$$2z_1 + 2z_2 + 2z_3 + z_4 = 11$$

Because $2z_1 + 2z_2 + 2z_3$ will give an even #, we know that z_4 must be odd

I am going to condition on z_4:

If $z_4 = 1$: $2z_1 + 2z_2 + 2z_3 = 10$

$z_1 + z_2 + z_3 = 5 \Rightarrow \left(\!\binom{3}{5}\!\right) = 21$

$z_4 = 3$ $2z_1 + 2z_2 + 2z_3 = 8$

$z_1 + z_2 + z_3 = 4 \Rightarrow \left(\!\binom{3}{4}\!\right) = 15$

$z_4 = 5$ $2z_1 + 2z_2 + 2z_3 = 6$

$z_1 + z_2 + z_3 = 3 \Rightarrow \left(\!\binom{3}{3}\!\right) = 10$

$z_4 = 7$ $2z_1 + 2z_2 + 2z_3 = 4$

$z_1 + z_2 + z_3 = 2 \Rightarrow \left(\!\binom{3}{2}\!\right) = 6$

$z_4 = 9$ $2z_1 + 2z_2 + 2z_3 =$

$z_1 + z_2 + z_3 = \Rightarrow \left(\!\binom{3}{1}\!\right) = 3$

$z_4 = 11$ $2z_1 + 2z_2 + 2z_3 = 0$

$z_1 + z_2 + z_3 = 0 \Rightarrow \left(\!\binom{3}{0}\!\right) = 1$

So, there are $21 + 15 + 10 + 6 + 3 + 1 =$ 56 possible solutions

(there has to be a better way for this?)