Homework #3 - Math 511

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#Math 511 /Unit 3

1) (5 points) Page 73 # 3 How many onto function $[8] \to [5]$ are possible? Give an exact numerical answer.

We first distribute the 8 distinct objects to 5 identical recipients, where each gets at least one. This is done in $S(8,5)$ ways. To label the recipients, there are $5!$ ways.

So, the # of onto functions from $[8] \to [5]$ is $S(8,5) \cdot 5! = 1050 \cdot 120 = 126{,}000$

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2) (5 points) Page 73 #7 Find the number of equivalence relations on a set.

Based on 1.4, we know that the number of equivalences relations is the same as the number of partitions. The Bell number $B(n)$ gives the number of partitions of an n-set.

So, the number of equivalence relations on an n-set is:

$$B(n) = \sum_{k=1}^{n} S(n,k)$$

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3) (10 points) Page 74 #11 Give a combinatorial proof: If $n \geq 1$ and $k \geq 1$, then

$$S(n,k) = \sum_{i=0}^{n-1} \binom{n-1}{i} S(i,k-1).$$

Pf: LHS Notice $S(n,k)$ Counts the number of ways to partition $[n]$ into k-blocks.

RHS Now, Condition on the number of elements that are not in the block Containing n.

If there are 0 elements not in the block w/ n, there are $\binom{n-1}{0}$ ways to choose them, and $S(0,k-1)$ ways to partition those elements into $k-1$ blocks

If there are 1 . . . . , there are $\binom{n-1}{1}$ . . . ., and $S(i,k-1)$ . . . .

⋮

If . . . . $n-1$ . . . . $\binom{n-1}{n-1}$ . . . ., and $S(n-1,k-1)$ . . . . . .

So, the number of elements that are not in the block Containing n is $\sum_{i=0}^{n-1} \binom{n-1}{i} S(i,k-1)$.

Thus, $S(n,k) = \sum_{i=0}^{n-1} \binom{n-1}{i} S(i,k-1)$ ∎

4)

(10 points) Page 74 #17 Solve a linear system to find number a, b, c, d, e so that the following polynomial equation is true:

$$x^4 = a(x)_0 + b(x)_1 + c(x)_2 + d(x)_3 + e(x)_4.$$

Here, $(x)_4 = x(x-1)(x-2)(x-3) = x^4 - 6x^3 + 11x^2 - 6x$ and $(x)_3 = x(x-1)(x-2) = x^3 - 3x^2 + 2x$, and so on, where $(x)_0 = 1$. Express the solution in terms of numbers studied in this section.

$x^4 = a(x)_0 + b(x)_1 + c(x)_2 + d(x)_3 + e(x)_4$

$(x)_4 = x(x-1)(x-2)(x-3) = x^4 - 6x^3 + 11x^2 - 6x$

$(x)_3 = x(x-1)(x-2) = x^3 - 3x^2 + 2x$

$(x)_2 = x(x-1) = x^2 - x$

$(x)_1 = x$

$(x)_0 = 1$

$x^4 = a + b(x) + c(x^2 - x) + d(x^3 - 3x^2 + 2x) + e(x^4 - 6x^3 + 11x^2 - 6x)$

$x^4 = a + bx + cx^2 - cx + dx^3 - 3dx^2 + 2dx + ex^4 - 6ex^3 + 11ex^2 - 6ex$

$x^4 = \underbrace{a}_{=0} + x\underbrace{(b - c + 2d - 6e)}_{=0} + x^2\underbrace{(c - 3d + 11e)}_{=0} + x^3\underbrace{(d - 6e)}_{=0} + x^4\underbrace{(e)}_{=1}$

So, we know: $e = 1$, because $1x^4 = 0 + 0x + 0x^2 + 0x^3 + ex^4$

Then, since $d - 6e = 0$, $d - 6(1) = 0$, $d = 6$ → $c - 3d + 11e = 0$, $c - 3(6) + 11(1) = 0$, $c - 7 = 0$, $c = 7$ → $b - c + 2d - 6e = 0$, $b - 7 + 2(6) - 6(1) = 0$, $b - 7 + 12 - 6 = 0$, $b - 1 = 0$, $b = 1$ → $a = 0$

So, $x^4 = 0(x)_0 + 1(x)_1 + 7(x)_2 + 6(x)_3 + 1(x)_4$

These values correspond to the Stirling numbers for $S(4,k)$ $0 \le k \le 4$.

Thus, $x^4 = \underbrace{S(4,0)}_{=a}\cdot(x)_0 + \underbrace{S(4,1)}_{=b}\cdot(x)_1 + \underbrace{S(4,2)}_{=c}\cdot(x)_2 + \underbrace{S(4,3)}_{=d}\cdot(x)_3 + \underbrace{S(4,4)}_{=e}\cdot(x)_4$

5) (12 points) Page 81 #1 You have 40 pieces of candy to distribute among 10 children. Find the number of ways to do this in each of the following situations. Leave your answer in standard notation.

(a) The pieces of candy are different and each child gets at least one piece.

$k=40$ distinct objects (candy) Recieve ≥ 1
$n=10$ distinct recipients (kids)

$S(40,10)\cdot 10!$

(b) The pieces of candy are indistinguishable and each child can get any number of pieces.

$k=40$ identical objects (candy) No Restrictions
$n=10$ distinct recipients (kids)

$\left(\!\binom{10}{40}\!\right)$

(c) The pieces of candy are different but you distribute them among 10 indistinguishable paper bags.

$k=40$ distinct objects (candy) No Restrictions
$n=10$ identical recipients (bags)

$\sum_{i=1}^{10} S(40,i)$

(d) The pieces of candy are indistinguishable but you distribute them among 10 indistinguishable paper bags and each bag contains at least on piece.

$k=40$ identical objects (candy) Recieves ≥ 1
$n=10$ identical recipients (bags)

$P(40,10)$

(e) The pieces of candy are different and each child gets exactly one piece, so there are some pieces left over.

$k=40$ distinct objects (candy) Recieves (1-1) exactly 1
$n=10$ distinct recipients (kids)

$(40)_{10}$ $\quad \frac{40}{1}\cdot\frac{39}{2}\cdot \dots \cdot \frac{31}{10}$

(f) The pieces of candy are different and Frank receives four pieces.

$k=40$ distinct objects (candy)
$n=10$ distinct recipients (9 kids + Frank)

$\underbrace{\binom{40}{4}}_{\text{Frank's pieces}}\cdot \underbrace{9^{36}}_{\text{No Restrictions on the other 9 kids}}$

6) (10 points) Page 82 #8 Give a bijective proof of the following: the number of partitions of n is equal to the number of partitions of $2n$ into n parts.

$$P(n) = P(2n, n)$$

$P(4)$ $n=4$ $P(8,4)$

$k=1$ $4 \to 5+1+1+1$

$3+1 \to 4+2+1+1$

$2+2 \to 3+3+1+1$

$2+1+1 \to 3+2+2+1$

$1+1+1+1 \to 2+2+2+2$

Let A be the set of partitions of n.

Let B be the set of partitions of $2n$ into n parts.

Let $f: A \to B$ be given by

$f(p) = q$, where q is obtained from p in the following way: Let k be the number of parts in p. If $k < n$, add $n-k$ parts of zero to p. Then increase each part by 1.

Well defined:

Let $f(p) = q$. q must be a partition of $2n$ into n parts. It must have n parts, because if p had $k < n$ parts, we added $n-k$ parts, and $n-k+k=n$. When we add 1 to each term, this ensures there are exactly n parts.
q must also be a partition of $2n$. If p is a partition of n, and we add one to each of the n parts, we have added n, and $n+n=2n$.

Inverse: f has an inverse. Let $q \in B$, & define $g(q) = p$, where we form p by reducing each part of q by 1 and removing any parts of size 0. Since we are removing 1 from n parts, $2n-n=n$, so p is a partition of n.

So, it is clear that f and g are inverses, so f is a bijection.

Thus, $|A| = |B|$. $|A| = P(n)$ and $|B| = P(2n, n)$. ∎

7) (10 points) How many integers are there from 1 to 10^{30} which are perfect squares, perfect cubes, or perfect 5^{th} powers?

Let $U = \{1, 2, \ldots, 10^{30}\}$.

$D_2 \subseteq U$ are all the perfect squares → Largest p.s. is $(10^{15})^2 = 10^{30}$ All p.s. $1^2, 2^2, 3^2, \ldots, (10^{15})^2$

$D_3 \subseteq U$ are all the numbers that are perfect cubes → Largest P.C is $(10^{10})^3 = 10^{30}$ P.C.: $1^3, 2^3, \ldots, (10^{10})^3$

$D_5 \subseteq U$ are all the numbers that are perfect 5^{th} powers → Largest 5th power is $(10^6)^5 = 10^{30}$ $1^5, 2^5, \ldots (10^6)^5$

Let d_2, d_3, d_5 be the properties that a number is a perfect square, cube, 5^{th} respectively.

We are finding $|D_2 \cup D_3 \cup D_5| = N_{\geq}(d_2) + N_{\geq}(d_3) + N_{\geq}(d_5) - N_{\geq}(d_2, d_3) - N_{\geq}(d_2, d_5) - N_{\geq}(d_3, d_5) + N_{\geq}(d_2, d_3, d_5)$

$N_{\geq}(d_2) = |D_2| = 10^{15}$ $N_{\geq}(d_3) = |D_3| = 10^{10}$ $N_{\geq}(d_5) = |D_5| = 10^6$

Now, also figure out overlap:

$D_2 \cap D_3$ are perfect $6^{ths} \to (10^5)^6$ is largest $|D_2 \cap D_3| = 10^5 = N_{\geq}(d_2, d_3)$

$D_2 \cap D_5$ are perfect $10^{ths} \to (10^3)^{10}$ is largest $|D_2 \cap D_5| = 10^3 = N_{\geq}(d_2, d_5)$

$D_3 \cap D_5$ are perfect $15^{th} \to (10^2)^{15}$ is largest $|D_3 \cap D_5| = 10^2 = N_{\geq}(d_3, d_5)$

$D_2 \cap D_3 \cap D_5$ are perfect $30^{ths} \to (10)^{30}$ is largest $|D_2 \cap D_3 \cap D_5| = 10 = N_{\geq}(d_2, d_3, d_5)$

Now, Substituting:

$$|D_2 \cup D_3 \cup D_5| = N_{\geq}(d_2) + N_{\geq}(d_3) + N_{\geq}(d_5) - N_{\geq}(d_2, d_3) - N_{\geq}(d_2, d_5) - N_{\geq}(d_3, d_5) + N_{\geq}(d_2, d_3, d_5)$$

$$= 10^{15} + 10^{10} + 10^6 - 10^5 - 10^3 - 10^2 + 10$$

$$= 1000010001000000 - 101100 + 10$$

$$= 1000010000898910$$ is the # of integers from 1 to 10^{30} that are a perfect square, cube, or 5^{th} power.

8) (20 points) Page 93 #5 After a day of skiing a family of six washes all of their ski-wear including their gloves. The next day, each family member grabs two gloves from the pile.

(a) Assume that everyone grabs one left-hand and one right-hand glove. In how many ways can they do this so that no one has both of their own gloves?

Let p_i be the property that person i gets their gloves
Let $P = \{p_1, p_2, \dots, p_6\}$

$N_{\geq}(\emptyset) = 6!\,6!$ (6! ways to distribute LHG, RHG) $= 518{,}400$

$N_{\geq}(p_i) = 5!\,5!$ (person i gets glove while rest are distributed however)

$N_{\geq}(p_i p_j) = 4!\,4!$

$\vdots$

$N_{\geq}(p_1 \cdots p_6) = 0!\,0! = 1$ (everyone gets Right & left gloves)

Since,

$$N_{=}(\emptyset) = \sum_{J : J \subseteq P} (-1)^{|J|} N_{\geq}(J)$$

$$= \sum_{|J|=0} N_{\geq}(J) - \sum_{|J|=1} N_{\geq}(J) + \dots + \sum_{|J|=6} N_{\geq}(J)$$

$$= \binom{6}{0} 6!\,6! - \binom{6}{1} 5!\,5! + \dots + \binom{6}{6} 0!\,0!$$

$$= \frac{6!\,\cancel{6!}\,6!}{0!\,\cancel{6!}} - \frac{6!\,\cancel{5!}\,5!}{\cancel{5!}\,1!} + \dots + \frac{6!}{0!\,6!} 0!\,0!$$

$$= \frac{6!\,6!}{0!} - \frac{6!\,5!}{1!} + \dots + \frac{6!}{6!}$$

$$D_6 = 6! \sum_{j=0}^{6} (-1)^j \left(\frac{(6-j)!}{j!} \right)$$

Now, Solving for D_6, using Maple $D_6 = 439{,}975$

(b) Answer part (a) assuming instead that each family member grabs any two gloves.

Let p_i be the property that person i gets their gloves
Let $P = \{p_1, p_2, \dots, p_6\}$

$$N_{\geq}(\emptyset) = \frac{12!}{2^6 \cdot 6!} \cdot 6!$$

12! ways to order gloves
$2^6 \to$ order within each pair
6! order of pairs
6! ways to assign pairs to people

$N_{\geq}(\emptyset) = \frac{12!}{2^6}$

$N_{\geq}(p_i) = \frac{10!}{2^5}$ (person i gets glove while rest are distributed however)

$N_{\geq}(p_i p_j) = \frac{8!}{2^4}$

$$N_{=}(\emptyset) = \sum_{J : J \subseteq P} (-1)^{|J|} N_{\geq}(J)$$

$$= \sum_{|J|=0} N_{\geq}(J) - \sum_{|J|=1} N_{\geq}(J) + \dots + \sum_{|J|=6} N_{\geq}(J)$$

$$= \binom{6}{0} \frac{12!}{2^6} - \binom{6}{1} \frac{10!}{2^5} + \binom{6}{2} \frac{8!}{2^4} - \dots + \binom{6}{6} \frac{0!}{2^0}$$

$$= \frac{6!}{0!\,6!} \cdot \frac{12!}{2^6} - \frac{6!}{1!\,5!} \frac{10!}{2^5} + \dots + \frac{6!}{6!\,0!} \cdot \frac{0!}{2^0}$$

This is Not a good why to simplif

$$D_6 = \sum_{j=0}^{6} (-1)^j \binom{6}{j} \frac{(12-2j)!}{2^{6-j}}$$

Now, Solving for D_6, using Maple $D_6 = 6{,}840{,}085$

9) (10 points) Prove by induction that for $n \geq 0$, $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$.

Base Case: Let $n=0$. $\sum_{k=0}^{0} k\binom{0}{k} \stackrel{?}{=} 0 \cdot 2^{0-1}$ Thus, the base case holds.

$$0\binom{0}{0} \stackrel{?}{=} 0 \cdot 2^{-1}$$

$$0 = 0 \checkmark$$

Assume that for some $n \geq 0$,

$$\sum_{k=0}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$$

Inductive Step: Now, prove $\sum_{k=0}^{n+1} k\binom{n+1}{k} = (n+1)2^{(n+1)-1}$

Using Pascal's identity: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

$$\sum_{k=0}^{n+1} k\left(\binom{n}{k-1} + \binom{n}{k}\right) = (n+1) \cdot 2^{n}$$

$$\underbrace{\sum_{k=0}^{n+1} k\binom{n}{k-1}} + \underbrace{\sum_{k=0}^{n+1} k\binom{n}{k}} = (n+1)2^{n}$$

Notice that

$$\sum_{k=0}^{n+1} k\binom{n}{k} = \sum_{k=0}^{n} k\binom{n}{k} + \underbrace{(n+1)\binom{n}{n+1}}_{\text{goes to zero}} = \sum_{k=0}^{n} k\binom{n}{k}$$

$$\sum_{k=0}^{n+1} k\binom{n}{k-1} = \sum_{k=1}^{n+1} k\binom{n}{k-1} \text{ because } 0\binom{n}{-1} = 0$$

Let $j = k-1$, (shift indices)

$$\sum_{j=0}^{n} (j+1)\binom{n}{j} = \sum_{j=0}^{n} j\binom{n}{j} + \sum_{j=0}^{n} \binom{n}{j}$$

$$\sum_{j=0}^{n} j\binom{n}{j} + \sum_{j=0}^{n} \binom{n}{j} + \sum_{k=0}^{n} k\binom{n}{k} = (n+1) \cdot 2^{n}$$

Now, by induction Hyp:

$$n \cdot 2^{n-1} + \sum_{j=0}^{n} \binom{n}{j} + n \cdot 2^{n-1} = (n+1) \cdot 2^{n}$$

By $\binom{n}{k}$ identity (2.3 pg 61)

$$n \cdot 2^{n-1} + 2^{n} + n \cdot 2^{n-1} = (n+1) \cdot 2^{n}$$

$$2n \cdot 2^{n-1} + 2^{n} = (n+1) \cdot 2^{n}$$

$$2^{n} \cdot n + 2^{n} = (n+1) \cdot 2^{n}$$

$$(n+1) \cdot 2^{n} = (n+1) \cdot 2^{n}$$

So, by induction, for $n \geq 0$,

$$\sum_{k=0}^{n} k\binom{n}{k} = n \cdot 2^{n-1} \blacksquare$$

10)

(10 points) Page 101 #3 Prove by induction that for $n \geq 2$ $\prod_{j=2}^{n}\left(1-\frac{1}{j^2}\right)=\frac{n+1}{2n}$.

Base Case: Let $n=2$. $\prod_{j=2}^{2}\left(1-\frac{1}{j^2}\right) \overset{?}{=} \frac{2+1}{2(2)}$ Thus, the base case holds.

$$\left(1-\frac{1}{2^2}\right) \overset{?}{=} \frac{3}{4}$$

$$\left(1-\frac{1}{4}\right) \overset{?}{=} \frac{3}{4}$$

$$\frac{3}{4}=\frac{3}{4} \checkmark$$

Assume that for some $n \geq 2$, $\prod_{j=2}^{n}\left(1-\frac{1}{j^2}\right)=\frac{n+1}{2n}$

Inductive Step: Now, prove $\prod_{j=2}^{n+1}\left(1-\frac{1}{j^2}\right)=\frac{(n+1)+1}{2(n+1)}$

$$\prod_{j=2}^{n+1}\left(1-\frac{1}{j^2}\right)=\frac{n+2}{2n+2}$$

Notice that $\prod_{j=2}^{n+1}\left(1-\frac{1}{j^2}\right)=\left(\prod_{j=2}^{n}\left(1-\frac{1}{j^2}\right)\right)\left(1-\frac{1}{(n+1)^2}\right)$

$$1-\frac{1}{(n+1)^2}=\frac{(n+1)^2-1}{(n+1)^2}=\frac{n^2+2n+1-1}{(n+1)^2}=\frac{n^2+2n}{(n+1)^2}$$

So, by inductive hypothesis:

$$\frac{n+1}{2n}\left(1-\frac{1}{(n+1)^2}\right)=\frac{n+2}{2n+2}$$

$$\frac{n+1}{2n}\cdot\frac{n^2+2n}{(n+1)^2}=\frac{n+2}{2n+2}$$

$$\frac{\cancel{n+1}}{2\cancel{n}}\cdot\frac{\cancel{n}(n+2)}{(n+1)^{\cancel{2}}}=\frac{n+2}{2n+2}$$

$$\frac{n+2}{2(n+1)}=\frac{n+2}{2n+2}$$

$$\frac{n+2}{2n+2}=\frac{n+2}{2n+2} \checkmark$$

So, by induction, for $n \geq 2$, $\prod_{j=2}^{n}\left(1-\frac{1}{j^2}\right)=\frac{n+1}{2n}$ ∎