Homework #4 Math 581

1) Pf

① We know that if $U=\emptyset$, $U\in\mathcal{T}$ by given parameters of the problem. Thus, $\emptyset\in\mathcal{T}$.

② We know that $X-X=\emptyset$. Since $\emptyset$ is finite, it then must also be countable. Thus $X\in\mathcal{T}$.

③ Let $U, V\in\mathcal{T}$. Then, $X-U$ and $X-V$ are countable.

Consider $X-(U\cap V)$. By DeMorgan's Law, $X-(U\cap V) = (X-U)\cup(X-V)$. By Thrm. 3.1.4 since both $(X-U)$ + $(X-V)$ are countable, their union is as well, proving that $U\cap V\in\mathcal{T}$.

④ Let $\{U_i \mid i\in I\}$ be a family of sets, where $U_i\in\mathcal{T}$ for each $i\in I$. Then $X-U_i$ is countable for each i.

Consider $X-\bigcup_{i\in I} U_i = \bigcap_{i\in I}(X-U_i)$. We know that $\bigcap_{i\in I}(X-U_i)$ must be a subset of X, because we cannot get more elements than the original set. Thus, by Thrm. 3.1.2, it must be countable. So, $\bigcup_{i\in I} U_i\in\mathcal{T}$.

Thus, since we have proven ①, ②, ③ + ④, $\mathcal{T}$ is a topology on X. ∎

---

3) $S: [0,1]\cup\{2\}\cup[3,4]$

Show 2 is a LP in $\mathbb{R}$ w/ finite Complement Topology.

Let U be an open neighborhood of 2. So, $\mathbb{R}-U$ is finite, or $U=\emptyset$.

We know that $U\neq\emptyset$, because $2\in U$ because it is an open neighborhood.

→ may be xtra info?

Since $\mathbb{R}-U$ is finite, there there are a finite amount of numbers not in U (by def)

We know S is infinite, because it contains both $[0,1]$ + $[3,4]$. So, we know that we are gaurenteed some points in S that are not in U, because U cannot cover all of S. So, the intersection $(U-\{2\})\cap S\neq\emptyset$.

* ?? please help!

$p \notin S$.

4) if p is not a L.P (limit point) iff, $\exists$ a open nb'hd U of p s.t. $U \cap S = \emptyset$.

($\Rightarrow$) Assume p is not a L.P. Let U be an open neighborhood of p, where $p \in U$. Since p is not a L.P., then $(U - \{p\}) \cap S = \emptyset$. Since we know that $p \notin S$, the $U \cap S = \emptyset$, proving that direction.

($\Leftarrow$) Now, Let U be an open neighborhood of p s.t. $U \cap S = \emptyset$. Thus, U and S contain none of the same points. So, by def, p is not a L.P because $p \notin S$, there are no points of S in U.

:)

---

5) $\mathbb{Z}_{\text{finit Complement topoly}}$

a) $\{1, 2, 3\}$ ↳ closed.

Complement is $\{\ldots, -2, -1, 0\} \cup \{4, 5, \ldots\}$ which is infinite, so $\{1,2,3\}$ is not open.

It's a finite set, so it must be closed.

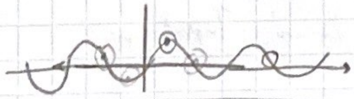
b) $\{p \mid p \text{ is a prime}\}$ ↳ neither

Complement is all composities, which is infinite so not open.

The set is infinite, not finite, so it's not closed.

$\mathbb{R}^2_{STD}$

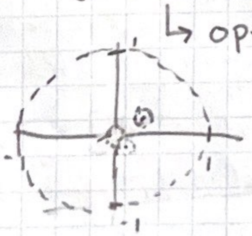
a) $\{(x, \sin(x)) \mid x \in \mathbb{R}\}$ ↳ Closed



Not open, Really easy to find open balls that contain points outside of $\sin(x)$

Closed, even if I pop out one small point, there are still other points in $\sin x$ in the open ball.

b) $\{(x,y) \mid 0 < x^2 + y^2 < 1\}$ ↳ open

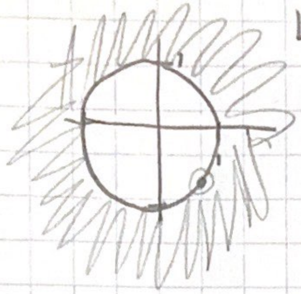


open, we can find open balls for any point in the donut-like object

not closed
the limit points would be the edges of the circle, or wher $x^2+y^2=1$. This is excluded though!

5c) $\{(x,y) \mid x^2+y^2 \geq 1$

↳ closed



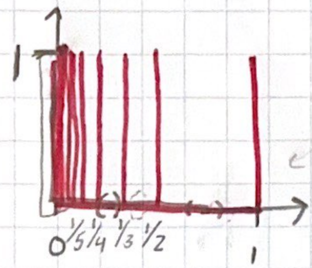
not open

Any of the points on the boundry $x^2+y^2=1$ will have a neighborhood w/ points not in the set.

Closed. I think the limit points are the unit circle $x^2+y^2=1$, which are definitly contained!

---

(6) $C = \{(x,0) \mid x \in [0,1]\} \cup \bigcup_{n=1}^{\infty} \{(\frac{1}{n}, y) \mid y \in [0,1]\}$



Limit Points. In the $\{(x,0) \mid x \in [0,1]\}$ section of the comb, every point is a limit point, because you can always find another $(x,0)$ in the neighborhood.
For the teeth of the Comb, just like in the sine Curve, we need to look @ where we get close to the y-axis. So those limit points are $\{(0,y) \mid y \in [0,1]\}$

Overall the limit Points are:

$$\{(x,0) \mid x \in [0,1]\} \cup \{(0,y) \mid y \in [0,1]\}$$

↑ This is alreay in C ↑ new part!

Closure → the set C plus the new Limit Point Set I added!

$$\overline{C} = C \cup \{(0,y) \mid y \in [0,1]\}$$