Homework #4 - Math 511

#Math 511 /Unit 4

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1) (4 points each) Page 113 #2 In each case find a concise OGF for answering the question and also identify the coefficient you need.

a) How many ways are there to put a total postage of 75 cents on an envelope, using 3-, 5-, 10-, and 12- cent stamps?

$$\underbrace{(1+x^3+x^6+x^9+x^{12}+\dots)}_{3¢\text{ stamp}}\underbrace{(1+x^5+x^{10}+x^{15}+x^{20}+\dots)}_{5¢\text{ stamp}}\underbrace{(1+x^{10}+x^{20}+x^{30}+\dots)}_{10¢\text{ stamp}}\underbrace{(1+x^{12}+x^{24}+x^{36}+\dots)}_{12¢\text{ stamp}}$$

$$\left(\frac{1}{1-x^3}\right)\cdot\left(\frac{1}{1-x^5}\right)\cdot\left(\frac{1}{1-x^{10}}\right)\cdot\left(\frac{1}{1-x^{12}}\right)$$

So, $G(x) = \dfrac{1}{(1-x^3)(1-x^5)(1-x^{10})(1-x^{12})}$, and we want to find the coeff. of x^{75}

b) At the movies you select 24 pieces of candy from 5 different types. How many ways can you do this if you want at least two pieces of each type?

$(x^2+x^3+x^4+\dots)^5$ ← 5 types of candy

↑ want at least 2 pieces

$$= \left(x^2(1+x+x^2+x^3+\dots)\right)^5 = x^{10}\underbrace{(1+x+x^2+\dots)^5}_{\frac{1}{(1-x)^5}}$$

↖ Factor out to create a geo series

So $G(x) = \dfrac{x^{10}}{(1-x)^5}$, and we need the coefficent of x^{24}.

However, we can just find the coeff. of x^{14} in $\dfrac{1}{(1-x)^5}$

c) How many solutions to $z_1 + z_2 + z_3 = 15$ are there, where the z_i are integers satisfying, $0 \le z_i \le 8$?

$$\underbrace{(1+x+x^2+x^3+\dots+x^8)}_{z_1}\underbrace{(1+x+x^2+x^3+\dots+x^8)}_{z_2}\underbrace{(1+x+x^2+x^3+\dots+x^8)}_{z_3}$$

$$= (1+x+x^2+x^3+\dots+x^8)^3 = \left(\frac{1-x^9}{1-x}\right)^3 = (1-x^9)^3\cdot\frac{1}{(1-x)^3}$$

So $G(x) = (1-x^9)^3\cdot\dfrac{1}{(1-x)^3}$, and we need the coeff. of x^{15}

d) How many ways are there to make change for a dollar using only pennies, nickels, dimes, and quarters?

$$(1+x+x^2+x^3+\dots)(1+x^5+x^{10}+x^{15}+\dots)(1+x^{10}+x^{20}+x^{30}+\dots)(1+x^{25}+x^{50}+x^{75}+\dots)$$

So, $G(x) = \dfrac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})}$, and we need the coeff. of x^{100}

2) Page 113 #3 Find the coefficient of...

a) x^k in $\frac{1+x+x^4}{(1-x)^5}$ $\qquad \frac{1+x+x^4}{(1-x)^5} = \frac{1}{(1-x)^5} + \frac{x}{(1-x)^5} + \frac{x^4}{(1-x)^5}$

$$\text{coeff. } x^k = \left(\!\binom{5}{k}\!\right) + \left(\!\binom{5}{k-1}\!\right) + \left(\!\binom{5}{k-4}\!\right)$$

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b) x^3 in $\frac{x}{(1-x)^8}$ $\qquad \frac{x}{(1-x)^8} = \left(\!\binom{8}{k-1}\!\right) \rightarrow$ coeff. of x^k

So, coefficent of $x^3 = \left(\!\binom{8}{2}\!\right) = 36$

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c) x^{50} in $(x^9 + x^{10} + x^{11} + \cdots)^3$

$$(x^9 + x^{10} + x^{11} + \ldots)^3$$
$$= (x^9(1 + x + x^2 + \ldots))^3$$
$$= x^{27}(1 + x + x^2 + \ldots)^3$$
$$= \frac{x^{27}}{(1-x)^3} = \left(\!\binom{3}{k-27}\!\right) \rightarrow \text{coeff. of } x^k$$

So, coeff. of $x^{50} = \left(\!\binom{3}{23}\!\right) = 300$

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d) x^{k-1} in $\frac{1+x}{(1-2x)^5} = \frac{1}{(1-2x)^5} + \frac{x}{(1-2x)^5}$

Start w/ $\frac{1}{(1-2x)^5} = \sum_{k=0}^{\infty} \left(\!\binom{5}{k}\!\right)(2x)^k \Rightarrow$ coeff. of x^k is $\left(\!\binom{5}{k}\!\right) \cdot 2^k$

Now $\frac{x}{(1-2x)^5} = \sum_{k=0}^{\infty} \left(\!\binom{5}{k}\!\right) \cdot (2x)^k \cdot x = \sum_{k=0}^{\infty} \left(\!\binom{5}{k-1}\!\right) 2^{k-1} x^{k+1} = \sum_{k=1}^{\infty} \left(\!\binom{5}{k-1}\!\right) 2^{k-1} x^k \Rightarrow$ Coeff of x^k is $\left(\!\binom{5}{k-1}\!\right) \cdot 2^{k-1}$

Thus, Coeff. of x^k is $\left(\!\binom{5}{k}\!\right) \cdot 2^k + \left(\!\binom{5}{k-1}\!\right) \cdot 2^{k-1}$

So, coeff. of $x^{k-1} \Rightarrow \left(\!\binom{5}{k-1}\!\right) \cdot 2^{k-1} + \left(\!\binom{5}{k-2}\!\right) \cdot 2^{k-2}$

3) (10 points) Page 124 #3 Use the convolution formula to find the coefficient of x^k in $\frac{(1+x)^n}{(1-x)^m}$.

$$\left[\!\left[\underbrace{(1+x)^n}_{\binom{n}{k}} \cdot \underbrace{\frac{1}{(1-x)^m}}_{\left(\!\binom{m}{k}\!\right)}\right]\!\right]_{x^k} = \boxed{\sum_{j=0}^{k} \binom{n}{j}\left(\!\binom{m}{k-j}\!\right)}$$

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4) Page 127 #10 Find the coefficient of $x^n/n!$ in $(e^x - 1)^3$.

$$\left[\!\left[e^{cx}\right]\!\right]_{\frac{x^k}{k!}} = c^k$$

$$\left[\!\left[(e^x-1)^3\right]\!\right]_{\frac{x^n}{n!}} = \left[\!\left[e^{3x} - 3e^{2x} + 3e^x - 1\right]\!\right]_{\frac{x^n}{n!}} = \underbrace{\left[\!\left[e^{3x}\right]\!\right]_{\frac{x^n}{n!}}}_{3^n} - \underbrace{\left[\!\left[3e^{2x}\right]\!\right]_{\frac{x^n}{n!}}}_{3\cdot 2^n} + \underbrace{\left[\!\left[3e^{x}\right]\!\right]_{\frac{x^n}{n!}}}_{3\cdot 1} - \underbrace{\left[\!\left[1\right]\!\right]_{\frac{x^n}{n!}}}_{\begin{cases}1 & \text{if } n=0\\ 0 & \text{if } n\geq 1\end{cases}}$$

$3^n - 3\cdot 2^n + 3\cdot 1 - \begin{cases}1 & \text{if } n=0\\ 0 & \text{if } n\geq 1\end{cases}$

If $n=0 \Rightarrow 3^0 - 3\cdot 2^0 + 3 - 1$
$1 - 3 + 3 - 1 = 0$

if $n \geq 1 \Rightarrow 3^n - 3\cdot 2^n + 3$

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5) Use generating functions to show that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 = \begin{cases} 0 & \text{if } n \text{ is odd;} \\ (-1)^m \binom{2m}{m} & \text{if } n = 2m. \end{cases}$$

We know $\sum_{k=0}^{n} \underbrace{\binom{n}{k}}_{b_j} x^k = (1+x)^n$ and $\sum_{k=0}^{n} \underbrace{(-1)^n \binom{n}{k}}_{a_j} x^k = (1-x)^n$

Let $f(x) = (1+x)^n$ and $g(x) = (1-x)^n$. So, $f(x)\cdot g(x) = (1+x)^n(1-x)^n = (1-x^2)^n$.

Now, expanding using the binomial Thrm, $(1-x^2)^n = \sum_{j=0}^{n} (-1)^n \binom{n}{j} x^{2j}$. We want to know the coefficent of x^n in $(1-x^2)^n$. This will only give even powers of x^n, since $2j$ will always give an even number.

So, if n is odd, the coeff. is 0. If n is even, or $n=2m$, coeff. of x^{2m} is $(-1)^m\binom{n}{m} = (-1)^m\binom{2m}{m}$.

Now, we need to show that $f(x)\cdot g(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 x^k$. Using the Convolution formula,

$$g(x)\cdot f(x) = \left(\sum_{k=0}^{n} \underbrace{(-1)^n \binom{n}{k}}_{a_j} x^k\right)\left(\sum_{k=0}^{n} \underbrace{\binom{n}{k}}_{b_j} x^k\right) = \sum_{k=0}^{n} (-1)^n \binom{n}{k} \underbrace{\binom{n}{n-k}}_{=\binom{n}{k}} = \sum_{k=0}^{n} (-1)^n \binom{n}{k}^2$$

So $\sum_{k=0}^{n} (-1)^n \binom{n}{k}^2 = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^m\binom{2m}{m} & \text{if } n = 2m \end{cases}$ ∎

6) (10 points) Page 132 #1d Solve the following recurrence relations using the generating function technique. $d_0 = 1$, $d_1 = 4$, and $d_n = 4d_{n-1} - 4d_{n-2}$ for $n \geq 2$.

Guess $= 2^n \cdot (n+1)$

n	d_n	
0	1	$2^0 \cdot 1$
1	4	$2^1 \cdot 2$
2	$4d_1 - 4d_0 = 4(4) - 4(1) = 12$	$2^2 \cdot 3$
3	$4d_2 - 4d_1 = 4(12) - 4(4) = 48 - 16 = 32$	$2^3 \cdot 4$
4	$4(32) - 4(12) = 128 - 48 = 80$	$2^4 \cdot 5$

$1 - 4x + 4x^2 = (1-2x)^2$

$F(x) = \frac{1}{(1-2x)^2} = \sum_{n=0}^{\infty} 2^n \left(\binom{2}{n}\right) x^n$

$2^n \binom{2+n-1}{n}$

$2^n \binom{n+1}{n}$

$2^n (n+1)$

① Let $F(x) = \sum_{n\geq 0} d_n x^n$

② $d_n x^n = (4d_{n-1} - 4d_{n-2}) x^n$

③ $\sum_{n\geq 2} d_n x^n = \sum_{n\geq 2} 4d_{n-1} x^n - \sum_{n\geq 2} 4d_{n-2} x^n$

④ $F(x) - d_1 x - d_0 = 4x \sum_{n\geq 1} d_n x^n - 4x^2 \sum_{n\geq 0} d_n x^n$

⑤ $F(x) - 4x - 1 = 4x(F(x) - d_0) - 4x^2(F(x))$

$F(x) - 4x - 1 = 4xF(x) - 4x - 4x^2 F(x)$

$F(x) - 4xF(x) + 4x^2 F(x) = 1$

$F(x)(1 - 4x + 4x^2) = 1$

$F(x) = \frac{1}{1 - 4x + 4x^2}$

$F(x) = \frac{1}{(1-2x)^2}$

⑥ $[F(x)]_{x^n} = 2^n (n+1)$

So $d_n = 2^n \cdot (n+1)$

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7) (10 points) Page 132 #2 Use an EGF to solve the recurrence relation $a_0 = 2$ and $a_n = na_{n-1} - n!$ for $n \geq 1$.

Guess: $n!(2-n)$

n	a_n	
0	2	$0! \cdot (2)$
1	$1 \cdot a_0 - 1! = 1 \cdot 2 - 1! = 2 - 1 = 1$	$1! \cdot (1)$
2	$2 \cdot a_1 - 2! = 2 - 2 = 0$	$2! \cdot (0)$
3	$3 \cdot a_2 - 3! = 3 \cdot 0 - 3! = -6$	$3! \cdot (-1)$
4	$4 \cdot a_3 - 4! = 4(-6) - 24 = -48$	$4! \cdot (-2)$

① Let $G(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$

② $a_n \frac{x^n}{n!} = (n a_{n-1} - n!) \frac{x^n}{n!}$

③ $\sum_{n\geq 1} a_n \frac{x^n}{n!} = \sum_{n\geq 1} a_{n-1} \cdot \frac{x^n}{(n-1)!} - \sum_{n\geq 1} n! \frac{x^n}{n!}$

④ $G(x) - a_0 = x \sum_{n\geq 1} a_{n-1} \frac{x^{n-1}}{(n-1)!} - \sum_{n\geq 1} x^n$

$x + x^2 + x^3 + \cdots$

$x(1 + x + x^2 + \cdots)$

$\frac{x}{1-x}$

$G(x) - a_0 = x \sum_{n\geq 0} a_n \frac{x^n}{n!} - \frac{x}{1-x}$

⑤ $G(x) - 2 = xG(x) - \frac{x}{1-x}$

$G(x)(1-x) = 2 - \frac{x}{1-x}$

5 cont: $G(x)(1-x) = \dfrac{2-2x-x}{1-x} = \dfrac{2-3x}{1-x}$

$G(x) = \dfrac{2-3x}{(1-x)^2} = \dfrac{2}{(1-x)^2} - \dfrac{3x}{(1-x)^2}$

$\left[\!\left[\dfrac{1}{(1-x)^2}\right]\!\right]_{x^n} = \sum_{k\geq 0} \left(\!\binom{2}{k}\!\right) x^n = \binom{2+k-1}{k} = \binom{k+1}{k} = k+1$

$\left[\!\left[\dfrac{1}{(1-x)^2}\right]\!\right]_{x^{n-1}} = \sum_{k\geq 0} (k+1)x^k = \sum_{k\geq 1} (k)x^{k-1}$

6) $\left[\!\left[G(x)\right]\!\right]_{\frac{x^n}{n!}} = 2\left[\!\left[\dfrac{1}{(1-x)^2}\right]\!\right]_{\frac{x^n}{n!}} - 3\left[\!\left[\dfrac{x}{(1-x)^2}\right]\!\right]_{\frac{x^n}{n!}}$

$= 2\cdot n!\left[\!\left[\dfrac{1}{(1-x)^2}\right]\!\right]_{x^n} - 3n!\left[\!\left[\dfrac{x}{(1-x)^2}\right]\!\right]_{x^n}$

$= 2\cdot n!(n+1) - 3n!\cdot n$

$= n!(2n+2-3n)$

So, $a_n = n!(2-n)$

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8) (10 points) Page 132 #6 Consider the recurrence relation given by $b_0 = 1$ and $b_n = \sum_{i=1}^{n} \frac{b_{n-i}}{i!}$ for $n \geq 1$. Let $B(x)$ be the OGF of $\{b_n\}_{n\geq 0}$. Show that $B(x) = \frac{1}{2-e^x}$.

n	b_n
0	1
1	$\frac{b_0}{1!} = \frac{1}{1} = 1$
2	$\frac{b_1}{1!} + \frac{b_0}{2!} = \frac{1}{1} + \frac{1}{2} = \frac{3}{2}$
3	$\frac{b_2}{1!} + \frac{b_1}{2!} + \frac{b_0}{3!} = \frac{3}{2} + \frac{1}{2} + \frac{1}{6} = \frac{13}{6}$
4	$\frac{b_3}{1!} + \frac{b_2}{2!} + \frac{b_1}{3!} + \frac{b_4}{4!} = \frac{13}{6} + \frac{3}{4} + \frac{1}{6} + \frac{1}{24} = \frac{85}{24}$

Let $B(x) = \sum_{n=0}^{\infty} b_n x^n$

Now, since $b_n = \sum_{i=1}^{n} \frac{b_{n-i}}{i!} = \sum_{i=1}^{n} \underbrace{\frac{1}{i!}}_{a_n} b_{n-i}$

convolution, but need index at zero

convolution of itself & $\frac{1}{n!}$?

Let $j = n-i$

$b_n = \sum_{j=0}^{n-1} \frac{1}{(n-j)!} b_j$

Now, We know that the convolution formula is

$$\left[\!\left[B(x)\cdot A(x)\right]\!\right]_{x^k} = \sum_{j=0}^{k} b_j a_{k-j}$$

We need to define $A(x) = \sum_{n\geq 0} a_n x^n$

and $a_n = \begin{cases} 0 & \text{if } n=0 \\ \frac{1}{n!} & \text{if } n\geq 1 \end{cases}$, since that is given for b_n.

So $A(x) = \sum_{n\geq 1} \frac{1}{n!} x^n \Rightarrow e^x - 1$

a_0 term

Since b_n is the convolution of itself and a_n when $n \geq 1$,

$B(x)\cdot A(x) = \sum_{n\geq 1} b_n x^n$

I feel like I may have done some math magic in this area...

$B(x) = \sum_{n=0}^{\infty} b_n x^n = \left(\sum_{n\geq 1} b_n x^n\right) + b_0$

$B(x) = \left(\sum_{n\geq 0} b_n x^n\right) + 1$
$\quad -1 \qquad\qquad\qquad -1$

$B(x) - 1 = \sum_{n\geq 1} b_n x^n$

$B(x)\cdot A(x) = B(x) - 1$

$B(x)\cdot(e^x - 1) = B(x) - 1$

$B(x)e^x - B(x) = B(x) - 1$

$1 = 2B(x) - B(x)e^x$

$1 = B(x)(2-e^x)$

$B(x) = \dfrac{1}{2-e^x}$ ■

9) (10 points) Let a_n be the sequence defined by $a_n = 2a_{n-1} + 8a_{n-2} - 9(n-2)$ for $n \geq 2$, where $a_0 = 1$ and $a_1 = 3$. Find a_n using generating functions.

n	a_n
0	1
1	3
2	$2a_1 + 8a_0 - 9(2-2) = 6 + 8 = 14$
3	$2a_2 + 8a_1 - 9(3-2) = 28 + 24 - 9 = 43$
4	$86 + 112 - 18 = 180$

① Let $F(x) = \sum_{n\geq 0} a_n x^n$

② $a_n x^n = 2a_{n-1} x^n + 8a_{n-2} x^n - 9(n-2)x^n$

$$\sum_{n\geq 2} a_n x^n = \sum_{n\geq 2} 2a_{n-1} x^n + \sum_{n\geq 2} 8a_{n-2} x^n - \sum_{n\geq 2} 9(n-2)x^n$$

$$F(x) - a_1 x - a_0 = 2x \sum_{n\geq 1} a_n x^n + 8x^2 \sum_{n\geq 0} a_n x^n - 9x^2 \sum_{n\geq 0} n\, x^n$$

$$F(x) - a_1 x - a_0 = 2x(F(x) - a_0) + 8x^2(F(x)) - 9x^2 \sum_{n\geq 0} n x^n$$

$$F(x) - 3x - 1 = 2xF(x) - 2x + 8x^2 F(x) - 9x^2\left(\frac{x}{(1-x)^2}\right)$$

$$F(x)(1 - 2x - 8x^2) = 3x + 1 - 2x - 9x^2\left(\frac{x}{(1-x)^2}\right)$$

$$F(x)(1 - 2x - 8x^2) = 1 + x - 9x^2\left(\frac{x}{(1-x)^2}\right)$$

$$= 1 + x - \frac{9x^3}{(1-x)^2}$$

$$F(x)(1-2x-8x^2) = \frac{1 - x - x^2 - 8x^3}{(1-x)^2}$$ ← Used Maple!

$$F(x) = \frac{1 - x - x^2 - 8x^3}{(1-x)^2(1-2x-8x^2)} = \frac{1 - x - x^2 - 8x^3}{(1-x)^2(1-4x)(1+2x)}$$ deg 3 / deg 4

Now, using Partial Fraction Decomp.,

$$F(x) = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1-4x} + \frac{D}{1+2x}$$

$$1 - x - x^2 - 8x^3 = A(1-x)(1-4x)(1+2x) + B(1-4x)(1+2x) + C(1-x)^2(1+2x) + D(1-x)^2(1-4x)$$

(For the next parts, I used a lot of Maple ☺)

Solve

Let $x = 1$

$$1-1-1^2-8\cdot 1^3 = 0 + B(1-4)(1+2) + 0 + 0$$
$$-9 = B(-9)$$
$$B = 1$$

Let $x = \frac{1}{4}$

$$1-\tfrac{1}{4}-\left(\tfrac{1}{4}\right)^2-8\left(\tfrac{1}{4}\right)^3 = 0 + 0 + C\left(1-\tfrac{1}{4}\right)^2\left(1+2\left(\tfrac{1}{4}\right)\right) + 0$$
$$\frac{9}{16} = \frac{27}{32}C$$
$$C = \frac{2}{3}$$

Let $x = -\frac{1}{2}$

$$1+\tfrac{1}{2}-\left(-\tfrac{1}{2}\right)^2-8\left(-\tfrac{1}{2}\right)^3 = 0 + 0 + 0 + D\left(1+\tfrac{1}{2}\right)^2\left(1-4\left(-\tfrac{1}{2}\right)\right)$$
$$\frac{9}{4} = \frac{27}{4}D$$
$$D = \frac{1}{3}$$

Now,

$$1-x-x^2-8x^3 = A(1-x)(1-4x)(1+2x) + B(1-4x)(1+2x) + C(1-x)^2(1+2x) + D(1-x)^2(1-4x)$$
$$1-x-x^2-8x^3 = A+B+C+D + (\underset{-1}{\sim})x + (\underset{-1}{\sim})x^2 + (\underset{-8}{\sim})x^3$$

So, $1 = A+B+C+D$

$$1 = A + 1 + \tfrac{2}{3} + \tfrac{1}{3}$$
$$1 = A + 2$$
$$A = -1$$

Now, Replacing A, B, C, D

$$F(x) = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1-4x} + \frac{D}{1+2x} \Rightarrow \frac{-1}{1-x} + \frac{1}{(1-x)^2} + \frac{2}{3}\cdot\frac{1}{1-4x} + \frac{1}{3}\cdot\frac{1}{1+2x}$$

$$[\![F(x)]\!]_{x^n} = \left[\!\left[\frac{-1}{1-x}\right]\!\right]_{x^n} + \left[\!\left[\frac{1}{(1-x)^2}\right]\!\right]_{x^n} + \left[\!\left[\frac{2}{3}\cdot\frac{1}{1-4x}\right]\!\right]_{x^n} + \left[\!\left[\frac{1}{3}\cdot\frac{1}{1+2x}\right]\!\right]_{x^n}$$

$$[\![F(x)]\!]_{x^n} = -1\cdot 1 + \left(\!\!\binom{2}{n}\!\!\right) + \frac{2}{3}\cdot 4^n + \frac{1}{3}\cdot(-2)^n$$
$$= -1 + (n+1) + \frac{2}{3}\cdot 4^n + \frac{1}{3}\cdot(-2)^n$$

$$a_n = n + \frac{2}{3}\cdot 4^n + \frac{1}{3}\cdot(-2)^n$$

Phew!! :)