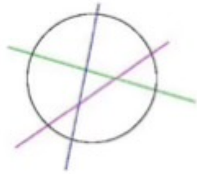
Homework # 5 - Math 626

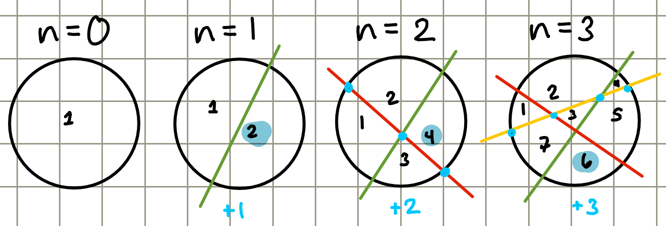
M[illegible]ehn #Math 626/Module 2

A1)

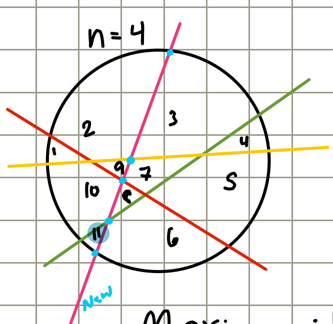
Start by drawing a circle [you can imagine this as a pizza]. We want to investigate cutting this circle into regions using line segments [imagine using a pizza cutter to cut along a single line]. Let n be the number of line segments we add to "cut" across the circle. Find the **maximum** number of regions for n = 4 and n = 5. Then, find an expression in terms of n that gives the maximum possible number of regions for n line segments. Below, you can see the n=3 case, which results in 7 regions.



For this problem, I started with simpler cases



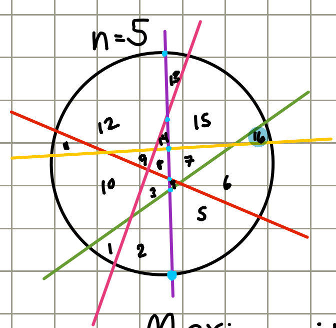
Every time, you add an additional segment, in order to make sure that you are getting the maximum number of regions, you need to make sure that the new segment intersects each previous segment.



Maximum: 11

↳ 7 regions previously

· The new chord intersects each previous chord, adding 4 regions



Maximum: 16

↳ 11 regions previously

· The new chord intersects each previous chord, adding 5 regions

n = segments	R(n) = Regions	
0	1	+1
1	2	+2
2	4	+3
3	7	+4
4	11	+5
5	16	
⋮		
n	R(n-1)	+n

So, $R(n) = R(n-1) + n$

The equation can be found recursively, by adding n to the previous regions. To find an explicit formula, I examined R(5) and solved it Recursively.

$R(5) = R(4) + 5$
$= R(3) + 4 + 5$
$= R(2) + 3 + 4 + 5$
$= R(1) + 2 + 3 + 4 + 5$
$= R(0) + 1 + 2 + 3 + 4 + 5$
$= 1 + 1 + 2 + 3 + 4 + 5$
$R(5) = 1 + \sum_{k=1}^{5} k$

$R(n) = 1 + \sum_{k=1}^{n} k$

(n+1) $\frac{n}{2}$ pairs

$= 1 + (1 + 2 + 3 + \ldots\ldots + n-2 + n-1 + n)$

$= 1 + \frac{n}{2}(n+1)$

$= 1 + \frac{n(n+1)}{2}$

$= \frac{n(n+1) + 2}{2}$

$R(n) = \frac{n^2 + n + 2}{2}$

Checks:

$R(4) = \frac{16 + 4 + 2}{2} = \frac{22}{2} = 11$ ✓

$R(5) = \frac{25 + 5 + 2}{2} = \frac{32}{2} = 16$ ✓

$R(2) = \frac{4 + 2 + 2}{2} = \frac{8}{2} = 4$ ✓

A2)

Suppose 6 numbered discs are placed on a grid of 7 squares as shown. Suppose that you can move any disc to an adjacent square, provided that square is unoccupied. Suppose you are also allowed to jump over ***a single disc*** in any direction into the square immediately past it (provided that square is empty). You are not allowed to jump over more than one disc [the only initial moves that are legal would be to move disc 1 one space to the left or to jump disc 2 over disc 1 into the empty square].

a) Find a sequence of legal moves that moves the discs from their present position (_,1,2,3,4,5,6) into the position (_,6,5,4,3,2,1) [note that the leftmost square should be empty in both the starting and ending position].

b) Suppose we add an additional square to the right and an additional disc with label 7. If possible, find a sequence of legal moves that moves the discs from the position (_1,2,3,4,5,6,7) to the "reversed" position (_,7,6,5,4,3,2,1).

Like most problems, I started with smaller values, in hopes of finding a pattern.

n=1

_	1	✓

0 moves

Doesn't help

n=2

_	1	2	
1	_	2	s←
1	2	_	s←
_	2	1	j→

3 moves

[s← / s← / j→

n=3

_	1	2	3	
2	1	_	3	j←
2	_	1	3	s→
2	3	1	_	j←
2	3	_	1	s←
_	3	2	1	j→

5 moves

j← / s→

j← / s←

j→

n=4

_	1	2	3	4	
1	_	2	3	4	s←
1	3	2	_	4	j←
1	3	2	4	_	s←
1	3	_	4	2	j→
_	3	1	4	2	j→
3	_	1	4	2	s←
3	4	1	_	2	j←
3	4	1	2	_	s←
3	4	_	2	1	j→
_	4	3	2	1	j→

10 moves

[s← / j← / s← / 2j→

s← / j← / s← / 2j→

n=5

_	1	2	3	4	5	
2	1	_	3	4	5	j←
2	_	1	3	4	5	s→
2	3	1	_	4	5	j←
2	3	1	4	_	5	s←
2	3	_	4	1	5	j→
_	3	2	4	1	5	j→
3	_	2	4	1	5	s←
3	4	2	_	1	5	j←
3	4	2	5	1	_	j←
3	4	2	5	_	1	s→
3	4	_	5	2	1	j→
_	4	3	5	2	1	j→
4	_	3	5	2	1	s←
4	5	3	_	2	1	j←
4	5	_	3	2	1	s→
_	5	4	3	2	1	j→

16 moves

j← / s→

j← / s←

2j→ / s←

2j← / s→

2j→ / s←

j← / s→

j→

I noticed that for when n=even, it helped to start with a slide. Additionally, all slides were to the left, and the jumps reversed. For n=4, the pattern was s← / 1j← / s← / 2j→ or $s/(\frac{n}{2})-1\,j / s / \frac{n}{2}\,j$, which matched the pattern for n=2. So, for n=6, I tried the same.

n=6

$s/(\frac{n}{2}-1)j / s / \frac{n}{2}j$

$s / 2j / s / 3j$

a)

_	1	2	3	4	5	6	
1	_	2	3	4	5	6	s←
1	3	2	_	4	5	6	j←
1	3	2	5	4	_	6	j←
1	3	2	5	4	6	_	s←
1	3	2	5	_	6	4	j→
1	3	_	5	2			j→
_	3	1	5	2	6	4	j→
3	_	1	5	2	6	4	s←
3	5	1	_	2	6	4	j←
3	5	1	6	2	_	4	j←
3	5	1	6	2	4	_	s←
3	5	1	6	_	4	2	j→
3	5	_	6	1	4	2	j→
_	5	3	6	1	4	2	j→
5	_	3	6	1	4	2	s←
5	6	3	_	1	4	2	j←
5	6	3	4	1	_	2	j←
5	6	3	4	1	2	_	s←
5	6	3	4	_	2	1	j→
5	6	_	4	3	2	1	j→
_	6	5	4	3	2	1	j→

21 moves

b) For n = odd, both started with the sequence (j / s / j / s), and ended with (j)

Then, for n = 5, there was a sequence of 3 reversing 2j / s along with one more j / s
Using this as a starting point, along with some guess and check, I ended with the following sequence:

n = 7

_	1	2	3	4	5	6	7	
2	1	_	3	4	5	6	7	j ←
2	_	1	3	4	5	6	7	s →
2	3	1	_	4	5	6	7	j ←
2	3	1	4	_	5	6	7	s ←
2	3	_	4	1	5	6	7	j →
_	3	2	4	1	5	6	7	j →
3	_	2	4	1	5	6	7	s ←
3	4	2	_	1	5	6	7	j ←
3	4	2	5	1	_	6	7	j ←
3	4	2	5	1	6	_	7	s ←
3	4	2	5	_	6	1	7	j →
3	4	_	5	2	6	1	7	j →
_	4	3	5	2	6	1	7	j →
4	_	3	5	2	6	1	7	s ←
4	5	3	_	2	6	1	7	j ←
4	5	3	6	2	_	1	7	j ←
4	5	3	6	2	7	1	_	j ←
4	5	3	6	2	7	_	1	s →
4	5	3	6	_	7	2	1	j →
4	5	_	6	3	7	2	1	j →
_	5	4	6	3	7	2	1	j →
5	_	4	6	3	7	2	1	s ←
5	6	4	_	3	7	2	1	j ←
5	6	4	7	3	_	2	1	j ←
5	6	4	7	_	3	2	1	s →
5	6	_	7	4	3	2	1	j →
_	6	5	7	4	3	2	1	j →
6	_	5	7	4	3	2	1	s ←
6	7	5	_	4	3	2	1	j ←
6	7	_	5	4	3	2	1	s →
_	7	6	5	4	3	2	1	j →

31 moves

I tried a few different ways, but the smallest number of moves I could find was 31.

j / s
j / s
2j / s
2j / s
3j / s
3j / s
3j / s
2j / s
2j / s
j / s
j

B2)

a) For what values of n is it possible to move from (_1,2,3,...,n-1,n) to (_n,n-1,...,3,2,1)? Justify your answer.

b) For those values of n that are possible, what is the minimum number of moves needed?

a) As demonstrated above, I found sequences for $n=2-7$. For $n=1$, the sequence is completed as _n is the same.
It is possible to find a sequence for any $n \geq 1$.

If we don't care about finding the minimum, just proving that a sequence exists, we can approach it in the following way:

_	1	2	3	...	n-1	n
1	_	2	3	...	n-1	n
1	2	_	3	...	n-1	n
1	2	3	_	...	n-1	n
1	2	3	...	_	n-1	n
1	2	3	...	n	n-1	_
1	2	3	...	n	_	n-1
1	2	3	n	_	...	n-1
1	2	_	n	3	...	n-1
1	2	n	_	3	...	n-1
1	_	n	2	3	...	n-1
1	n	_	2	3	...	n-1
_	n	1	2	3	...	n-1
n	_	1	2	3	...	n-1
n	n-1	...	3	2	1	_

Slide discs until _ n-1 n

jump n over n-1

- jump over n,
- Slide n,
- repeat until n is in the first square

Repeat until reversed

Slide all discs over one at a time.

This proves that for any $n \geq 1$, there is a possible sequence.

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b) When determining the minimum number of moves, like part A2, it helps to break it into odd and even cases.

<u>Even</u>

<u>n=2</u> : 3

S / S / j

$1(1+0+1+1)$

$1(\underline{3})$ n+1

<u>n=4</u> : 10

S / j / S / 2j

S / j / S / 2j

$2(1+1+1+2)$

$2(\underline{5})$ n+1

<u>n=6</u>: 21

S / 2j / S / 3j

S / 2j / S / 3j

S / 2j / S / 3j

$3(1+2+1+3)$

$3(\underline{7})$ n+1

n = even:

S / $(\frac{n}{2}-1)$j / S / $(\frac{n}{2})$j

Repeat $\frac{n}{2}$ times

$\frac{n}{2}(n+1)$

<u>Check:</u>

n=8 → see below

So, when n=even, the number of moves can be found by $M(n) = \frac{n(n+1)}{2}$

Check $n=8$:

_	1	2	3	4	5	6	7	8	
1	_	2	3	4	5	6	7	8	s←
1	3	2	_	4	5	6	7	8	j←
1	3	2	5	4	_	6	7	8	j←
1	3	2	5	4	7	6	_	8	j←
1	3	2	5	4	7	6	8	_	s←
1	3	2	5	4	7	_	8	6	j→
1	3	2	5	_	7	4	8	6	j→
1	3	_	5	2	7	4	8	6	j→
_	3	1	5	2	7	4	8	6	j→

3	_	1	5	2	7	4	8	6	s←
3	5	1	_	2	7	4	8	6	j←
3	5	1	7	2	_	4	8	6	j←
3	5	1	7	2	8	4	_	6	j←
3	5	1	7	2	8	4	6	_	s←
3	5	1	7	2	8	_	6	4	j→
3	5	1	7	_	8	2	6	4	j→
3	5	_	7	1	8	2	6	4	j→
_	5	3	7	1	8	2	6	4	j→

5	_	3	7	1	8	2	6	4	s←
5	7	3	_	1	8	2	6	4	j←
5	7	3	8	1	_	2	6	4	j←
5	7	3	8	1	6	2	_	4	j←
5	7	3	8	1	6	2	4	_	s←
5	7	3	8	1	6	_	4	2	j→
5	7	3	8	_	6	1	4	2	j→
5	7	_	8	3	6	1	4	2	j→
_	7	5	8	3	6	1	4	2	j→

7	_	5	8	3	6	1	4	2	s←
7	8	5	_	3	6	1	4	2	j←
7	8	5	6	3	_	1	4	2	j←
7	8	5	6	3	4	1	_	2	j←
7	8	5	6	3	4	1	2	_	s←
7	8	5	6	3	4	_	2	1	j→
7	8	5	6	_	4	3	2	1	j→
7	8	_	6	5	4	3	2	1	j→
_	8	7	6	5	4	3	2	1	j→ ✓

36 moves

$M(8) = \frac{8(9)}{2} = \frac{72}{2} = 36$ ✓

To make the equation easier, think about as k's.

$n=3 \to k=1$

$n=5 \to k=2$

$n=7 \to k=3$

$n \to k=\boxed{\frac{n-1}{2}}$

Odd $\boxed{n \geq 3}$

$n=3$: 5

$k=1$, $3 \cdot k$

j / s
j / s
j

$(1 \cdot 3) + 2$

jumps + slides

$(k \cdot 3) + 2$

$n=5$: 16

$k=2$, $3 \cdot k$

j / s
j / s
2j / s
2j / s
2j / s
j / s
j

$(1 \cdot 2) + (2 \cdot 3) + (1 \cdot 2) + 6$ slides

$((k-1) \cdot 2) + (k \cdot 3) + ((k-1) \cdot 2) + 6$

$n=7$: 31

$k=3$, $3 \cdot k$

j / s
j / s
2j / s
2j / s
3j / s
3j / s
3j / s
2j / s
2j / s
j / s
j

$(1 \cdot 2) + (2 \cdot 2) + (3 \cdot 3) + (2 \cdot 2) + (1 \cdot 2) + 10$ slide

$((k-2) \cdot 2) + ((k-1) \cdot 2) + (k \cdot 3) + ((k-1) \cdot 2) + ((k-2) \cdot 2) + 10$

$n =$ odd

$(1 \cdot 2) + (2 \cdot 2) + \ldots + ((k-1) \cdot 2) + (k \cdot 3) + ((k-1) \cdot 2) + \ldots + (2 \cdot 2) + (1 \cdot 2) + \text{slides}$

$4(1) + 4(2) + \ldots + 4(k-1) + 3k + \text{slides}$

$4 \sum_{i=1}^{k-1} i + 3k + \text{slides}$

$4 \frac{(k-1)k}{2} + 3k + \text{slides}$

$2k(k-1) + 3k + \text{slides}$

$2k(k-1) + 3k + 4k - 2$

$2k^2 - 2k + 3k + 4k - 2$

$= 2k^2 + 5k - 2$

$= 2\left(\frac{n-1}{2}\right)^2 + 5\left(\frac{n-1}{2}\right) - 2$

$= \frac{(n-1)^2}{2} + \frac{5(n-1)}{2} - \frac{4}{2}$

$= \frac{n^2 - 2n + 1 + 5n - 5 - 4}{2}$

$= \frac{n^2 + 3n - 8}{2}$

Slides - This is an arithmetic sequence

$2, 6, 10, \ldots$ (+4, +4)

$= 2 + (k-1) \cdot 4$

$= 2 + 4k - 4$

Slides $= 4k - 2$

So, when $n =$ odd, $M(n) = \frac{n^2 + 3n - 8}{2}$

Check: $n=9$:

_	1	2	3	4	5	6	7	8	9	
2	1	_	3	4	5	6	7	8	9	j ←
2	_	1	3	4	5	6	7	8	9	s →
2	3	1	_	4	5	6	7	8	9	j ←
2	3	1	4	_	5	6	7	8	9	s ←
2	3	_	4	1	5	6	7	8	9	j →
_	3	2	4	1	5	6	7	8	9	j →
3	_	2	4	1	5	6	7	8	9	s ←
3	4	2	_	1	5	6	7	8	9	j ←
3	4	2	5	1	_	6	7	8	9	j ←
3	4	2		1	6	_	7	8	9	s ←
3	4	2	5	_	6	1	7	8	9	j →
3	4	_	5	2	6	1	7	8	9	j →
_	4	3	5	2	6	1	7	8	9	j →
4	_	3	5	2	6	1	7	8	9	s ←

4	5	3	_	2	6	1	7	8	9	j ←
4	5	3	6	2	_	1	7	8	9	j ←
4	5	3	6	2	7	1	_	8	9	j ←
4	5	3	6	2	7	1	8	_	9	s ←
4	5	3	6	2	7	_	8	1	9	j →
4	5	3	6	_	7	2	8	1	9	j →
4	5	_	6	3	7	2	8	1	9	j →
_	5	4	6	3	7	2	8	1	9	j →
5	_	4	6	3	7	2	8	1	9	s ←
5	6	4	_	3	7	2	8	1	9	j ←
5	6	4	7	3	_	2	8	1	9	j ←
5	6	4	7	3	8	2	_	1	9	j ←
5	6	4	7	3	8	2	9	1	_	j ←
5	6	4	7	3	8	2	9	_	1	s →
5	6	4	7	3	8	_	9	2	1	j →
5	6	4	7	_	8	3	9	2	1	j →
5	6	_	7	4	8	3	9	2	1	j →
_	6	5	7	4	8	3	9	2	1	j →
6	_	5	7	4	8	3	9	2	1	s ←

6	_	5	7	4	8	3	9	2	1	
6	7	5	_	4	8	3	9	2	1	j ←
6	7	5	8	4	_	3	9	2	1	j ←
6	7	5	8	4	9	3	_	2	1	j ←
6	7	5	8	4	9	_	3	2	1	s →
6	7	5	8	_	9	4	3	2	1	j →
6	7	_	8	5	9	4	3	2	1	j →
_	7	6	8	5	9	4	3	2	1	j →
7	_	6	8	5	9	4	3	2	1	s ←
7	8	6	_	5	9	4	3	2	1	j ←
7	8	6	9	5	_	4	3	2	1	j ←
7	8	6	9	_	5	4	3	2	1	s →
7	8	_	9	6	5	4	3	2	1	j →
_	8	7	9	6	5	4	3	2	1	j →
8	_	7	9	6	5	4	3	2	1	s ←
8	9	7	_	6	5	4	3	2	1	j ←
8	9	_	7	6	5	4	3	2	1	s →
_	9	8	7	6	5	4	3	2	1	j →

50 moves

$$M(9) = \frac{n^2+3n-8}{2} = \frac{81+27-8}{2} = \frac{100}{2} = 50 \checkmark$$

So, the formula to find the minimum number of moves is:

$$M(n) = \begin{cases} \dfrac{n(n+1)}{2} ; & n \text{ is even} \\ \dfrac{n^2+3n-8}{2} ; & n \text{ is odd} \end{cases} \Bigg\} \text{ for } n \geq 2$$
$$\qquad 0 \quad ; \; n=1$$

A3) Suppose the three sides of a triangle have lengths a, b, and c respectively. Further suppose that a, b, and c are positive integers with $a \le b \le c$.

a) Given that c = 7, find the number of different triangles that can be formed by assigning appropriate values to a and b.
b) Given that c = 8, find the number of different triangles that can be formed by assigning appropriate values to a and b.
c) Given that c = 9, find the number of different triangles that can be formed by assigning appropriate values to a and b.

We know that by the triangle inequality thrm., the sum of the lengths of any 2 sides must be greater than the third side. So, $a+b>c$, $a+c>b$, and $b+c>a$. Since c is the longest side, we only have to look at $a+b>c$.

a) $c=7$

$a+b>7$ $(a \le b \le c)$

1, 7	2, 6	3, 5	4, 4
2, 7	3, 6	4, 5	
3, 7	4, 6	5, 5	
4, 7	5, 6		
5, 7	6, 6		
6, 7	~~7, 6~~ ← can't work because $b \ge a$		
7, 7			

$7+5+3+1=$ 16 triangles

b) $c=8$

$a+b>8$

1, 8	2, 7	3, 6	4, 5
2, 8	3, 7	4, 6	5, 5
3, 8	4, 7	5, 6	
4, 8	5, 7	6, 6	
5, 8	6, 7		
6, 8	7, 7		
7, 8			
8, 8			

$8+6+4+2=$ 20 triangles

c) $c=9$

$a+b>9$

1, 9	2, 8	3, 7	4, 6	5, 5
2, 9	3, 8	4, 7	5, 6	
3, 9	4, 8	5, 7	6, 6	
4, 9	5, 8	6, 7		
5, 9	6, 8	7, 7		
6, 9	7, 8			
7, 9	8, 8			
8, 9				
9, 9				

$9+7+5+3+1=$ 25 triangles

B3) a) Find a general rule that represents the number of different triangles that can be formed in terms of that value of c.
b) How do your answers to A3 parts a), b) and c) and B3 part a) change if we insist that a, b, and c all have distinct values?

Again, it's helpful to look at simpler cases.

c=1: 1, 1 — 1

c=2: 1, 2; 2, 2 — 2

c=3: 1, 3; 2, 3; 3, 3; 2, 2 — 4

c=4: 1, 4; 2, 4; 3, 4; 4, 4; 2, 3; 3, 3 — 6

c=5: 1, 5; 2, 5; 3, 5; 4, 5; 5, 5; 2, 4; 3, 4; 4, 4; 3, 3 — 9

c=6: 1, 6; 2, 6; 3, 6; 4, 6; 5, 6; 6, 6; 2, 5; 3, 5; 4, 5; 5, 5; 3, 4; 4, 4 — 12

Table for c's:

c	T(c)	
1	1	• Perfect Square
2	2	
3	4	• PS
4	6	
5	9	• PS
6	12	
7	16	• PS
8	20	
9	25	• PS

Then I examined even values.

There seemed to be perfect squares with the odd values, so I focused on those.

Turn into k

$k=1 \Rightarrow n=1 \rightarrow 1 = 1^2 = k^2$
$k=2 \Rightarrow n=3 \rightarrow 4 = 2^2 = k^2$
$k=3 \Rightarrow n=5 \rightarrow 9 = 3^2 = k^2$
$k=4 \Rightarrow n=7 \rightarrow 16 = 4^2 = k^2$
$k=5 \Rightarrow n=9 \rightarrow 25 = 5^2 = k^2$

$n=2k-1$
$n+1=2k$
$\boxed{\frac{n+1}{2}=k}$

$T(n)=\left(\frac{n+1}{2}\right)^2$ for n = odd

$k=1 \Rightarrow n=2 \rightarrow 2 = 1\cdot 2 = k(k+1)$
$k=2 \Rightarrow n=4 \rightarrow 6 = 2\cdot 3 = k(k+1)$
$k=3 \Rightarrow n=6 \rightarrow 12 = 3\cdot 4 = k\cdot(k+1)$
$k=4 \Rightarrow n=8 \rightarrow 20 = 4\cdot 5 = k\cdot(k+1)$

$n=2k$
$\boxed{k=\frac{n}{2}}$

$T(n)=\left(\frac{n}{2}\right)\left(\frac{n}{2}+1\right)$
$=\left(\frac{n}{2}\right)\left(\frac{n+2}{2}\right)$

$T(n)=\frac{n(n+2)}{4}$ for n = even

So, to find the # of triangles based on the length of the third side c:

$$T(c) = \begin{cases} \dfrac{(c+1)^2}{4} & \text{for } c = \text{odd} \\[2ex] \dfrac{c\,(c+1)}{4} & \text{for } c = \text{even} \end{cases}$$

---

b) If we insist that a, b, c are all distinct, it helps to write out all cases again for $c = 1 \to 9$

$c=1$: ~~1,1~~ ($a=b=c$) → 0

$c=2$: ~~1,2~~, ~~2,2~~ ($b=c$) → 0

$c=3$: ~~1,3~~, ~~2,3~~, ~~3,3~~ ($b=c$); ~~2,2~~ ($a=b$) → 0

$c=4$: ~~1,4~~, ~~2,4~~, ~~3,4~~, ~~4,4~~ ($b=c$); 2,3; ~~3,3~~ ($a=b$) → 1

$c=5$: ~~1,5~~, ~~2,5~~, ~~3,5~~, ~~4,5~~, ~~5,5~~; 2,4; 3,4; ~~4,4~~; ~~3,3~~ → 2

$c=6$: ~~1,6~~, ~~2,6~~, ~~3,6~~, ~~4,6~~, ~~5,6~~, ~~6,6~~; 2,5; 3,5; 4,5; 3,4; ~~4,4~~; ~~5,5~~ → 4

$c=7$: ~~1,7~~, ~~2,7~~, ~~3,7~~, ~~4,7~~, ~~5,7~~, ~~6,7~~, ~~7,7~~; 2,6; 3,6; 4,6; 5,6; 3,5; 4,5; ~~5,5~~; ~~6,6~~; ~~4,4~~ → 6

$c=8$: ~~1,8~~, ~~2,8~~, ~~3,8~~, ~~4,8~~, ~~5,8~~, ~~6,8~~, ~~7,8~~, ~~8,8~~; 2,7; 3,7; 4,7; 5,7; 6,7; 3,6; 4,6; 5,6; 4,5; ~~5,5~~; ~~6,6~~; ~~7,7~~ → 9

$c=9$: ~~1,9~~, ~~2,9~~, ~~3,9~~, ~~4,9~~, ~~5,9~~, ~~6,9~~, ~~7,9~~, ~~8,9~~, ~~9,9~~; 2,8; 3,8; 4,8; 5,8; 6,8; 7,8; 3,7; 4,7; 5,7; 6,7; 4,6; 5,6; ~~5,5~~; ~~6,6~~; ~~7,7~~; ~~8,8~~ → 12

Table for C's:

c	T'(c)
1	0
2	0
3	0
4	1
5	2
6	4
7	6
8	9
9	12

So, the values are in the same order, just shifted by 3.

$$T(c) = T'(c+3)$$

To write more explicitly, I again focused on odds vs. even: $\longrightarrow$

$$T'(c) = \begin{cases} \dfrac{c^2-4c+3}{4} & \text{for } c = \text{odd} \\[2ex] \dfrac{(c-2)^2}{4} & \text{for } c = \text{even} \end{cases}$$

	new $T'(c)$	c	old $T(c)$
$1\cdot 0$ $(k-1)k$	0	1	1
$0\cdot 1$	0	3	4
$1\cdot 2$	2	5	9
$2\cdot 3$	6	7	16
$3\cdot 4$	12	9	25
	⋮	⋮	
		c	$\frac{(c+1)^2}{4}$

$k = \frac{c-1}{2}$

$$\left(\frac{c-1}{2}-1\right)\left(\frac{c-1}{2}\right) = \left(\frac{c-1-2}{2}\right)\left(\frac{c-1}{2}\right) = \left(\frac{c-3}{2}\right)\left(\frac{c-1}{2}\right) = \boxed{\frac{c^2-4c+3}{4}}$$

So $T'(c) = \dfrac{c^2-4c+3}{4}$ for $c =$ odd

	new $T'(c)$	c	old $T(c)$
$(k-1)^2$ 0^2	0	2	2
$(k-1)^2$ 1^2	1	4	6
$(k-1)^2$ 2^2	4	6	12
3^2	9	8	20
	⋮	⋮	⋮
	$\boxed{\frac{(c-2)^2}{4}}$	c	$\frac{c\,(c+2)}{4}$

$k = \frac{c}{2}$

$$\left(\frac{c}{2}-1\right)^2 = \left(\frac{c-2}{2}\right)^2 = \frac{(c-2)^2}{4}$$

So $T'(c) = \dfrac{(c-2)^2}{4}$ for $c =$ even

A4)

Consider the numbers 1, 2, and 8. There are six permutations involving these three digits (without repetition): 128, 182, 218, 281, 812, and 821. If we add these six three-digit numbers together and take their average, we get 2442/6 = 407.

Notice that:

- the average ends up being an integer
- the digit sum of both the original digits and the digits in the average are equal: 1 + 2 + 8 = 4 + 0 + 7

a) Let a, b, and c be a set of three distinct positive one-digit integers. Determine whether or not the average of the six permutations of these three integers must always be an integer.

b) In cases where the average ends up being an integer, determine whether or not the sum of the original digits and the sum of the digits in the average will always end up being equal.

a) First, we know for a 3 digit integer, there a 6 permutations of the digits. To find the sum of the permutations, we know that the 100s digit is $100 \cdot d$, 10s digit is $10 \cdot d$, and 1s digit is d.

Perms:			
a	b	c	$\rightarrow 100a + 10b + c$
a	c	b	$100a + 10c + b$
b	a	c	$100b + 10a + c$
b	c	a	$100b + 10c + a$
c	a	b	$100c + 10a + b$
c	b	a	$100c + 10b + a$

Sum: $222a + 222b + 222c$
$= 222(a+b+c)$

Average: $\frac{222(a+b+c)}{6} = 37(a+b+c)$

Since a, b, c are integers, we know that a+b+c is an integer, so $37(a+b+c)$ is also an integer.

---

b) Now, we want to determine if a+b+c matchs the digit sum of $37(a+b+c)$.

When I looked up the exact definition for digit sum I came across 2 different definitions

1) Add each digit once: $2+5+9=16$
2) Add each digit until there is only one digits : $2+5+9=16 \Rightarrow 1+6=7$
   ↳ Also called repeated digit sum or digital root

Exploring each defintion to determine if there is a difference

Def 1: $124 \rightarrow 37(1+2+4) \rightarrow \overset{\text{Average}}{259}$ $1+2+4 \overset{?}{=} 2+5+9$
$7 \neq 16$

By definition 1, the sum of the original digits is not always equal to the sum of the average digits.

Def 2: $124 \rightarrow 37(1+2+4) \rightarrow 259$ $1+2+4 = 2+5+9$
$7 = \underset{1+6}{16}$
$7 = 7$ ✓

$289 \rightarrow 37(2+8+9) \rightarrow 703$ $2+8+9 = 7+0+3$
$19 = 10$
$1+9 = 1+0$
$10 = 1$
$1+0 = 1$
$1 = 1$ ✓

Since $37 \% 9 = 1$
(37 mod 9)

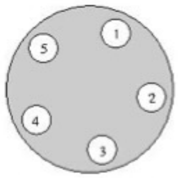
We know that
$(37 \cdot k) \% 9 = k \% 9$
$37 \% 9 \cdot k \% 9 = k \% 9$
$1 \cdot \underbrace{k \% 9} = \underbrace{k \% 9}$ ✓

By definition 2, the sum of the original digits is always equal to the sum of the average digits.

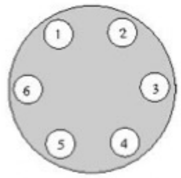
A5)

A "Simplex Lock" is a mechanical lock that opens only when the correct "combination" (a sequence of button pushes) is entered. The rules for setting a "combination" are as follows:

- A "combination" is a sequence of button pushes, with each push involving ***at least one*** button.
- Each button may be used ***at most once*** (once you press any button, it stays pushed in).
- Each push may include any of the buttons that have not been pushed yet, up to and including all remaining buttons (so a single "push" can involve pressing more than one button simultaneously).
- The "combination" does not need to use all of the buttons.
- When two or more buttons are pressed at the same time (as part of the same push), since they are pressed simultaneously, the order does not matter.



A 5 Button Simplex Lock



A 6 Button Simplex Lock

Here are a few possible "combinations" for the 5 Button Simplex Lock:

- {{1,2,4}, {3,5}} [this represents first pressing buttons 1, 2, and 4 simultaneously, then 3 and 5 simultaneously]
- {{3,5}, {1,2,4}} [this represents first pressing buttons 3 and 5 simultaneously, then 1, 2 and 4 simultaneously] Note that these first two examples are NOT the same combination.
- {{2}, {5}, {1}} [this represents pushing button 2 by itself, then 5 by itself, and then 1 by itself]
- {{ }} [this represents leaving it unlocked]
- {{1,2,3,4,5}} [this represents pressing all 5 buttons at the same time]

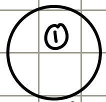
a) How many distinct "combinations" are there on a 5-button Simplex Lock?
b) How many distinct "combinations" are there on a 6-button Simplex Lock?
c) How many buttons are needed to give at least 1 million different "combinations"?

- **Before** discussing the problem with your group, spend 40-45 minutes working on this problem *by yourself*.
- For the first 10 minutes, write down a list of different strategies that could be used to help solve the problem and any mathematical concepts and resources that might be necessary. Keep a record of the ideas that you generate.
- Shortly thereafter, spend an additional 30-35 minutes working to make progress on solving the problem. Once this time is up, stop and note how far you got in your solution attempt.
- Later, after you have had a chance to make significant progress working on the problem with your group, write a paragraph comparing and contrasting what it was like working on this problem by yourself vs. working on this problem with your group.
- **You should turn in notes on your 10-minute "brainstorming session", a record of the progress you made during your 30-minute individual work session, and your paragraph reflection with your other standard solutions when you submit the individual portion of this Problem Set.**

10-minute brainstorming:

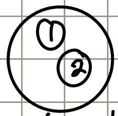
- Start w/ 2, 3, 4 simplex locks
  - ↳ pattern?
- Must be something w/ permutations/Combinations
  - ↳ Something Combinatorics?
    - Maybe partitions?
      - ↳ How many ways to divide a set!!
- Systematic way of Counting?
- Maybe recursion?
  - ↳ ... → 5 → 4 → 3 → 2 → 1
- Stirling #'s } Can't remember exact definitions
- Bell #'s

30-35 mins:



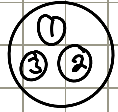
- ∅
- 1

2

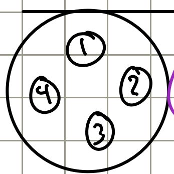


- ∅ · 1/2
- 1 · 2/1
- 2 · 12

6



· ∅	1/2	2/1	3/1	12/3
1	1/3	2/3	3/2	13/2
2	1/23	2/13	3/12	23/1
3	1/2/3	2/1/3	3/1/2	
12	1/3/2	2/3/1	3/2/1	
13				
23		8+15+3 = 26		
123				



∅	12	123	1234
1	13	124	
2	14	134	
3	23	234	
4	24		
	34		

Partitions ?

Start w/ 1:

1/2	1/2/3	1/2/4	1/2/34	1/2/3/4	1/2/4/3
1/3	1/3/2	1/3/4	1/3/24	1/3/2/4	1/3/4/2
1/4	1/4/2	1/4/3	1/4/23	1/4/2/3	1/4/3/2
1/23	1/23/4				
1/24	1/24/3				
1/34	1/34/2				
1/234					

7+6+12
= 25

25×4

start w/ 1-4

12/3 12/4 12/34 12/3/4 12/4/3

5 × 6
= 30

Start w/ pair of 2

123/4

1×4
= 4

start w/ 3#s

= 16 + 100 + 30 + 4

= 150

5:

1	12	123	1234	12345
2	13	124	1235	∅
3	14	125	1245	
4	15	134	1345	
5	23	135	2345	
	24	145		
	25	234		
	34	235		
	35	245		
	45	345		

1: 1/(1#) 1/(1#)/(2#) 1/(1#)/(3#) 1/(2#) 1/(2#)/(1#) 1/(2#)/(1#)/(1#) 1/(2#)/(2#) 1/(3#) 1/(3#)/(1#) 1/(4#)

1/(1#)	1/(1#)/(2#)	1/(1#)/(3#)	1/(2#)	1/(2#)/(1#)	1/(2#)/(1#)/(1#)	1/(2#)/(2#)	1/(3#)	1/(3#)/(1#)	1/(4#)
4	$\underbrace{4}$ × 3	$\underbrace{4}$ × 1	$\underbrace{6}$	$\underbrace{6}$ ×2	$\underbrace{6}$ ×2	$\underbrace{6}\ \underbrace{1}$	4	$\underbrace{4}$ ×1	1
	12	4		12	12	6	1/234	4	
	1/2/34 1/2/45 1/2/35								

Times-Up, I know I'm missing some

Paragraph Reflection:

Working on this problem by myself first allowed me to think about what it specifically was asking. I was able to break down my thinking a bit more, and work through some simpler versions of the problem. However, there got to be a point when I was working through the problem (specifically the 5-button case), that there was just too much, and it was getting tough to organize my thinking.

When I started working with my group, I was able to bring up some ideas that I had considered but hadn't spent a good deal of time with. We talked about partitions and Bell Numbers as being a possibility, but it didn't seem like that was a good path to follow, as it seemed like we could probably break it down as a combination and think about it as patterns. It was helpful to talk through the reasoning with others and get immediate feedback. It also helped me find what patterns that I was missing when I was trying to break down the 5-button problem.