#Math 511 /Unit 6

Homework #6 - Math 511

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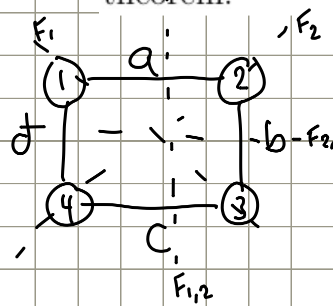
1) Page 205 #3 In how many different ways can we construct a square using four sticks and four styrofoam balls, where each stick is either black or white? (That is, we are coloring edges and not corners.)

a) (8 points) Label the square's edges a, b, c, d. Find the cycle structure of each element of the dihedral group as it acts on the edges and then apply the CFB theorem.



This is still D_4, so all of the same symmetries would still apply.

π	Product of Cycles	$\lvert fix_{D_4}(\pi)\rvert$
I	$(a)(b)(c)(d)$	2^4
R_{90}	$(a\,b\,c\,d)$	2
R_{180}	$(a\,c)(b\,d)$	2^2
R_{270}	$(a\,d\,c\,b)$	2^1
$F_{1,2}$	$(a)(b\,d)(c)$	2^3
$F_{2,3}$	$(a\,c)(b)(d)$	2^3
F_2	$(a\,b)(c\,d)$	2^2
F_1	$(a\,d)(b\,c)$	2^2

Because each cycle can be white or black.

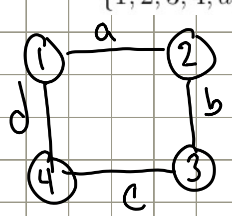
Using CFB, $|O| = \frac{1}{|G|}\sum_{\pi \in G} |fix_G(\pi)|$

$|O| = \frac{1}{8}(16+2+4+2+8+8+4+4)$

$|O| = 6$

So, there are 6 colorings of the Sticks.

b) (8 points) Now change the question to: In how many different ways can we construct a square using four sticks and four styrofoam balls, where each stick is either black or white and each ball is either black or white? Answer this using the CFB theorem. (Hint: The dihedral group now acts on the set $\{1, 2, 3, 4, a, b, c, d\}$ of corners and edges.



π	Product of Cycles	$\lvert fix_{D_4}(\pi)\rvert$
I	$(a)(b)(c)(d)(1)(2)(3)(4)$	2^8
R_{90}	$(a\,b\,c\,d)(1\,2\,3\,4)$	2^2
R_{180}	$(a\,c)(b\,d)(1\,3)(2\,4)$	2^4
R_{270}	$(a\,d\,c\,b)(1\,4\,3\,2)$	2^2
$F_{1,2}$	$(a)(b\,d)(c)(1\,2)(3\,4)$	2^5
$F_{2,3}$	$(a\,c)(b)(d)(1\,4)(2\,3)$	2^5
F_2	$(a\,b)(c\,d)(1\,3)(2)(4)$	2^5
F_1	$(a\,d)(b\,c)(1)(2\,4)(3)$	2^5

Using CFB: $|O| = \frac{1}{8}(256+4+16+4+32+32+32+32)$

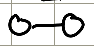
$= \frac{1}{8}(408)$

$|O| = 51$, So there a 51 colorings!

2) Let p be a prime number. This problem will walk you through a proof of Fermat's Little Theorem, which says that c^p and c have the same remainder when divided by p. (That is $c^p \equiv c \pmod{p}$.)

a) (8 points) Consider a necklace with p beads. Let two necklaces be equivalent if one can be obtained from another via rotation. That is consider the subgroup of D_p that only consists of the identity and the $p-1$ non-trivial rotations. Find the number of non-equivalent colorings of the necklace using c colors.

$p=2$

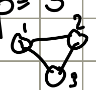


I	$(1)(2)$	c^2
R_1	(12)	c^1

$\frac{1}{p}(c+c^2)$

$\frac{1}{2}(c+c^2)$

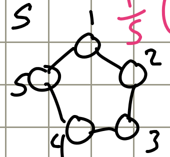
$p=3$



I	$(1)(2)(3)$	c^3
R_1	(123)	c^1
R_2	(132)	c^1

$\frac{1}{3}(c^3+2c)$

$p=5$ $\quad \frac{1}{5}(c^5+4c)$



I		c^5
R_1	(12345)	c^1
R_2	(13524)	c^1
R_3	(14253)	c^1
R_4	(15432)	c^1

p $\quad I \to c^p$

$R_1 \cdots R_{p-1} \to (p-1)c^1$

Because p is prime, each non-Identity Rotation will consist of a single cycle.

$\frac{1}{p}(c^p+(p-1)c)$

For each p, the Identity rotation will have p 1-cycles, so the $\mathrm{fix}_{D_p}(I)=c^p$, since there are c colors to choose for each cycle.

For each non-trivial rotation, each product of cycles will be a p-cycle, because there are no smaller cycles, because p is prime.

So $|\Theta| = \frac{1}{p}\left(c^p + (p-1)\cdot c\right)$

P-1 Rotations + Identity

P-1 Rotations

b) (4 points) Notice that the number of non-equivalent colorings must be an integer. Use this fact and your result from part (a) to show that $\frac{1}{p}c^p - \frac{1}{p}c$ is an integer.

We know that $|\Theta|$ must be an integer.

We can rewrite $|\Theta|$: $\frac{c^p+(p-1)c}{p} = \frac{c^p+cp-c}{p} = \frac{c^p-c+cp}{p} = \frac{1}{p}c^p - \frac{1}{p}c + c.$

So, $|\Theta| = \left(\frac{1}{p}c^p - \frac{1}{p}c\right) + c.$

Since $|\Theta|$ is an integer, and c is an integer, then it follows that $\left(\frac{1}{p}c^p - \frac{1}{p}c\right)$ must also be an integer.

c) (4 points) Why does this mean that c^p and c have the same remainder when divided by p?

Because $\frac{1}{p}c^p - \frac{1}{p}c$ is an integer, this means that $c^p - c$ is divisible by p, so $c^p - c \equiv 0 \pmod{p}$.

Then, if $c^p \equiv m \pmod{p}$ and $c \equiv n \pmod{p}$, so $c^p - c = m - n$. Since $m-n=0$, then $m=n$.

So $c^p \equiv c_n \pmod{p}$, they must have the same remainder when divided by p.

3) (16 points) Page 213 #11 Repeat the problem of counting the black-white (we did black and red in the video) colorings of the 3×3 grid, but do so for the 4×4 and the 5×5 grid.

4x4 Grid

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

4	8	12	16
3	7	11	15
2	6	10	14
1	5	9	13

π	Product of Cycles	$fix_a(\pi)$
I	(1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11)(12)(13)(14)(15)(16)	2^{16}
R_{90}	(1 4 16 13)(2 8 15 9)(3 12 14 5)(6 7 11 10)	2^4
R_{180}	(1 16)(2 15)(3 14)(4 13)(5 12)(6 11)(7 10)(8 9)	2^8
R_{270}	(1 13 16 4)(2 9 15 8)(3 5 14 12)(6 10 11 7)	2^4

$$|\mathcal{O}| = \frac{1}{4}(2^{16} + 2^4 + 2^8 + 2^4)$$
$$= \frac{1}{4}(65824) = 16{,}456$$

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

25	24	23	22	21
20	19	18	17	16
15	14	13	12	11
10	9	8	7	6
5	4	3	2	1

π	Product of Cycles	$fix_a(\pi)$
I	(1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11)(12)(13)(14)(15)(16)(17)(18)(19)(20)(21)(22)(23)(24)(25)	2^{25}
R_{90}	(1/5/25/21)(2/10/24/16)(3/15/23/11)(4/20/22/6)(7/9/19/17)(8/14/18/12)(13)	2^7
R_{180}	(1/25)(2/24)(3 23)(4 22)(5 21)(6 20)(7 19)(8 18)(9 17)(10 16)(11 15)(12 14)(13)	2^{13}
R_{270}	(1/21/25/5)(2/16/24/10)(3/11/23/15)(4/6/22/20)(7/17/19/9)(8/12/18/14)(13)	2^7

$$|\mathcal{O}| = \frac{1}{4}(2^{25} + 2^7 + 2^{13} + 2^7)$$
$$= \frac{1}{4}(33{,}562{,}880) = 8{,}390{,}720$$

4) (12 points) Page 213 #12 Generalize the previous problem to count the k-colorings of the $n \times n$ grid.

number of cycles

Identity operation: There will be n^2 cycles

R_{90} / R_{270}: These two operations will produce the same $|fix_a(\pi)|$

We need to condition on if n is odd or even.

- If n is even, each cell will form a 4-cycle, so there are $\frac{n^2}{4}$ cycles.
- If n is odd, there will be one fixed square exactly in the middle.

So, there are $1 + \frac{n^2-1}{4} = \frac{4+n^2-1}{4} = \frac{n^2+3}{4}$ cycles

R_{180}: Again, condition on if n is odd or even

- If n is even, each cell will form a 2-cycle, so there are $\frac{n^2}{2}$ cycles.
- If n is odd, there will be one fixed square exactly in the middle.

So, there are $1 + \frac{n^2-1}{2} = \frac{2+n^2-1}{2} = \frac{n^2+1}{2}$

So, for k-colorings, each $|fix_a(\pi)| = k^{\#\text{ of cycles}}$

IF n is odd:

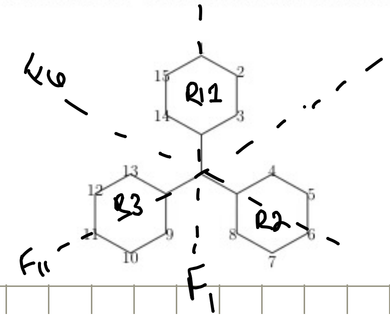
$$|\mathcal{O}| = \frac{1}{4}\left(k^{n^2} + 2 \cdot k^{\frac{n^2+3}{4}} + k^{\frac{n^2+1}{2}}\right)$$

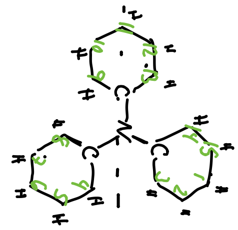
IF n is even:

$$|\mathcal{O}| = \frac{1}{4}\left(k^{n^2} + 2 \cdot k^{\frac{n^2}{4}} + k^{\frac{n^2}{2}}\right)$$

5)

The molecule triphenylamine has three rings of six carbon atoms attached to a central nitrogen atom, as drawn below, and 15 hydrogen atoms, with one attached to each carbon atom except the three carbons attached to the central nitrogen atom.





Flip

a)

(6pts) Find the group of symmetries of this molecule as a subgroup of $\mathcal{S}_{15}$. There are 6 elements in this group.

The 6 elements in this group are: Identity (I), Rotate 120° (R_1), Rotate 240° (R_2) Flip along the hydrogen atom #1, Swap Rings w/ atoms 4-8 with atoms 9-13 and swap atom 2+3 w/ 14+15 (F_1)
Similarly, Flip along hydrogen atom #6 (F_6) and hydrogen atom #11 (F_{11})

	π	Cycles of hydrogen atoms.	Cycle Index	$\lvert fix_a(\pi)\rvert$
	I	(1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11)(12)(13)(14)(15)	$(z_1)^{15}$	3^{15}
120°	R_1	(1 6 11)(2 7 12)(3 8 13)(4 9 14)(5 10 15)	$(z_3)^5$	3^5
240°	R_2	(1 11 6)(2 12 7)(3 13 8)(4 14 9)(5 15 10)	$(z_3)^5$	3^5
	F_1	(1)(2 15)(3 14)(4 13)(5 12)(6 11)(7 10)(8 9)	$(z_1)^1(z_2)^7$	3^8
	F_6	(1 11)(2 10)(3 9)(4 8)(5 7)(6)(12 15)(13 14)	$(z_1)^1(z_2)^7$	3^8
	F_{11}	(1 6)(2 5)(3 4)(7 15)(8 14)(9 13)(10 12)(11)	$(z_1)^1(z_2)^7$	3^8

b)

Find the cycle index of this group.

$$Z(z_1, z_2, z_3) = \frac{1}{6}\left(z_1^{15} + 2z_3^{5} + 3z_1 z_2^{7}\right)$$

c)

(3pts) How many isomers can be formed by replacing six of the hydrogen atoms with hydroxyl groups (OH)? (Use a pattern inventory.)

We will let Hydrogen $= h$ and Hydroxyl groups $(OH) = x$.

Replace z_k with $h^k + x^k$ in the cycle index to produce the pattern inventory

$$Z(z_1, z_2, z_3) = \frac{1}{6}\left(z_1^{15} + 2z_3^{5} + 3z_1 z_2^{7}\right)$$

$$P_G(h,x) = Z(h+x, h^2+x^2, h^3+x^3) = \frac{1}{6}\left((h+x)^{15} + 2(h^3+x^3)^5 + 3(h+x)(h^2+x^2)^7\right)$$

$$[\![P_G(h,x)]\!]_{x^6h^9} = 855$$ such isomers using Maple.

d) (3pts) How many isomers can be formed by replacing five of the hydrogen atoms with methyl groups (CH_3) and another five of the hydrogen atoms with fluorine atoms? (Use a pattern inventory.)

We will let Hydrogen $= h$, methyl groups $(CH_3) = m$, fluorine $= f$

Replace Z_k with $h^k + m^k + f^k$ in the cycle index to produce the pattern inventory

$$Z(z_1, z_2, z_3) = \frac{1}{6}\left(z_1^{15} + 2z_3^{5} + 3z_1 z_2^{7}\right)$$

$$P_G(h,m,f) = Z(h+m+f,\, h^2+m^2+f^2,\, h^3+m^3+f^3) = \frac{1}{6}\left((h+m+f)^{15} + 2(h^3+m^3+f^3)^5 + 3(h+m+f)(h^2+m^2+f^2)^7\right)$$

$[\![P_G(h,m,f)]\!]_{h^5m^5f^5} = 126{,}126$ such isomers using Maple

e) Suppose now that you are just coloring the corners with the hydrogen atoms with the colors red, white and black. Use a pattern inventory to answer the following questions.

We will let red $= r$, white $= w$, black $= b$

Replace Z_k with $r^k + w^k + b^k$ in the cycle index to produce the pattern inventory

$$Z(z_1, z_2, z_3) = \frac{1}{6}\left(z_1^{15} + 2z_3^{5} + 3z_1 z_2^{7}\right)$$

$$P_G(r,w,b) = Z(r+w+b,\, r^2+w^2+b^2,\, r^3+w^3+b^3) = \frac{1}{6}\left((r+w+b)^{15} + 2(r^3+w^3+b^3)^5 + 3(r+w+b)(r^2+w^2+b^2)^7\right)$$

i) (3pts) How many nonequivalent colorings have each color used the same number of times?

$$[\![P_G(r,w,b)]\!]_{r^5b^5w^5} = 126{,}126$$

ii) (3pts) How many nonequivalent colorings have no red?

$$[\![P_G(r,w,b)]\!]_{r^0} = b^{15} + 3b^{14}w + 21b^{13}w^2 + 81b^{12}w^3 + 238b^{11}w^4 + 511b^{10}w^5$$
$$+ 855b^9w^6 + 1090b^8w^7 + 1090b^7w^8 + 855b^6w^9 + 511b^5w^{10} + 238b^4w^{11}$$
$$+ 81b^3w^{12} + 21b^2w^{13} + 3bw^{14} + w^{15}$$

Adding all coefficents, we get 5600 colorings

iii) (3pts) How many nonequivalent colorings have at least one red?

To find the total # of colorings, I found $|\Theta|$, the number of 3-colorings.

$$|\Theta| = \frac{1}{6}(3^{15} + 2\cdot 3^5 + 3\cdot 3^8) = 2{,}394{,}846 = N_{Total} \text{ (number of total colorings)}$$

$$N_{\geq 1 Red} = N_{Total} - N_{0 Red} = 2{,}394{,}846 - 5600 = 2{,}389{,}246$$ colorings with at least one red.

iv) (3pts) How many nonequivalent colorings have 5 red, 6 black and 4 white?

$$[\![P_G(r,w,b)]\!]_{r^5b^6w^4} = 105210$$

6) (12 points) Page 236 #7 Prove that if $\delta(G) \geq k$, then G contains a path of length at least k.

Pf: Assume that the longest path, P, in G has a length of ℓ, where $\ell < k$.

Because P has a length of ℓ, there must be $\ell+1$ vertices, called $v_1, v_2, \ldots, v_\ell, v_{\ell+1}$

For each $v_i \in P$, $\deg(v_i) \leq \ell$, because v_i can connect to at most ℓ other verticies. If v_i could connect to more than ℓ vertices, P could be extended to a longer path.

We also know that since $\delta(G) \geq k$, then the degree of every vertex in P must be at least k, so $\deg(v_i) \geq k$.

So, $\deg(v_i) \leq \ell$ and $\deg(v_i) \geq k$, so $k \leq \deg(v_i) \leq \ell \Rightarrow k \leq \ell$, however, this contradicts our assumption.

Thus, if $\delta(G) \geq k$, then G must contain a path of length at least k. ∎