# Homework #7 - Math 511

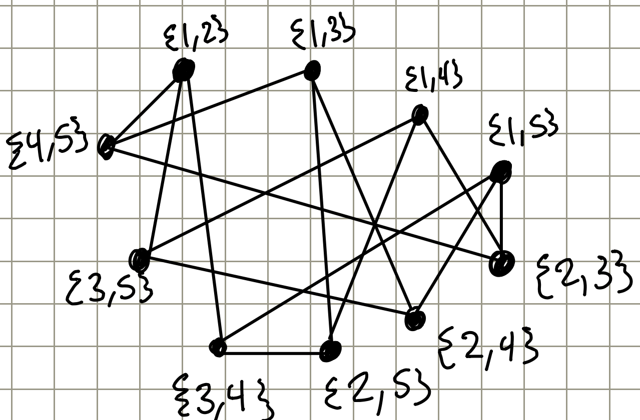
Maggie Kuehn

1) (12 points) Page 236 #2 Let $G$ be a graph whose vertex set is the set of 2-subsets of [5] and where two vertices are adjacent if and only if their corresponding subsets are disjoint

$[5] = \{1,2,3,4,5\}$

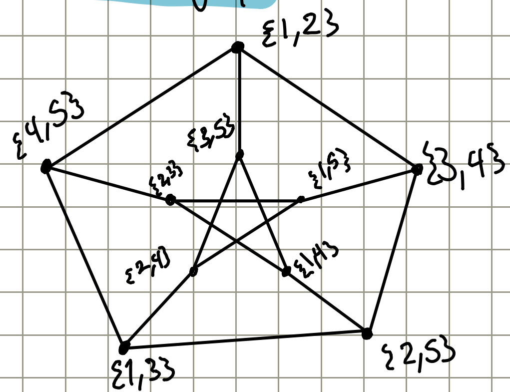
$V = \{\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}\}$

a) Draw $G$.

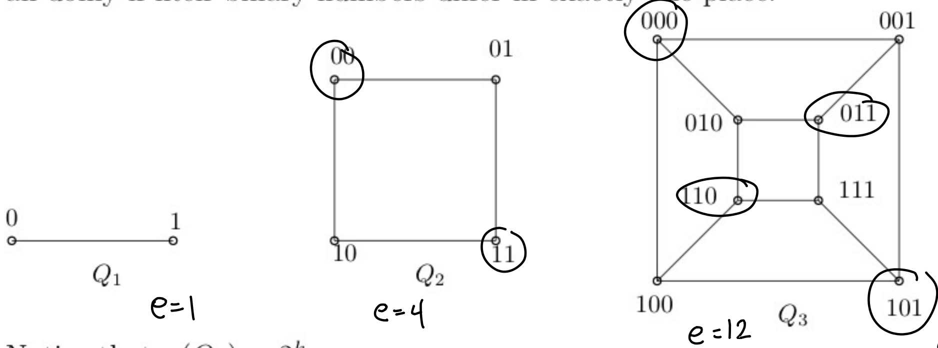



b) Find, with proof, a graph mentioned in this section that is isomorphic to $G$. (A picture counts as a proof if you give a little explanation to go with it.)

This is a graph with 10 vertices and is 3-regular. Because of this, I believe that G is isomorphic to the Petrson graph



2) (10 points) Page 237 #9 For $k \geq 1$, the graph $Q_k$ is called the $k$-dimensional cube. It's vertex set is the set of $k$-digit binary numbers, and two vertices are adjacent if an donly if hteir binary numbers differ in exactly one place.




Notice that $n(Q_k) = 2^k$.

0000
0001
0010
0100
1000
all adjacent

00000
00001
00010
00100
01000
10000
5

## a) Find $e(Q_k)$.

Let $v$ be a vertex of $Q_k$. This means that $v$ is a $k$-digit binary number.
We know there are exactly $k$ neighbors, because there are $k$-bits there are $k$ options for 1 bit to flip.
So, $\deg(v) = k$ for all $v$ in $Q_k$.

Using the handshaking lemma, we know that

$$\sum_{v \in V(Q_k)} \deg(v) = 2 \cdot e(Q_k) \Rightarrow \underbrace{2^k}_{\#\text{ of vertices}} \cdot \underbrace{k}_{\deg\text{ of each}} = 2 \cdot e(Q_k)$$

$$\frac{k \cdot 2^k}{2} = e(Q_k)$$

$$k \cdot 2^{k-1} = e(Q_k)$$

So, $e(Q_k) = k \cdot 2^{k-1}$

## b) Prove that $Q_k$ is bipartite, for all $k \geq 1$.

Let $V_1 = \{v \in V(Q_k) \mid v$ has an even # of 1's in the binary number$\}$
and $V_2 = \{v \in V(Q_k) \mid v$ has an odd # of 1's in the binary number$\}$

To prove that a graph is bipartite, each edge of the graph must have one endpoint in each block.
Let $u, v \in V_1$ and $u \neq v$. So, $u$ and $v$ both have an even number of 1's.

If $\{u, v\} \in e(Q_k)$, then they differ by one bit. However, with connecting $\{u, v\}$, we would need to flip 2 bits. $\boxed{\text{even \# 1's} \xrightarrow[\text{flip 1 bit}]{} \text{odd \# 1's} \xrightarrow[\text{flip 1 bit}]{} \text{even \# 1's}}$

So $\{u, v\} \notin e(Q_k)$. Similarly, this is true for all $v \in V_2$.
//
Let $u \in V_1$. All possible adjacent vertices of $u$ will differ by one bit, either adding one more or one less 1 in the binary representation. So, all endpoints will have an odd # of 1's.

So any $u \in V_1$ will have it's adajacent vertex in $V_2$. Similarly, this is true for all $v \in V_2$.
//
Because all edges connect a vertex in $V_1$ to $V_2$, $Q_k$ must be bipartite. ∎

3) (10 points) Page 237 #10a Use graphs to give a combinatorial proof of

$$\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}.$$

LHS

Notice $\binom{n}{2}$ counts the number of edges in an $K_n$ graph.

RHS

Now, we are going to partition the $n$ vertices into a set with $k$ vertices, and a set with $n-k$ vertices.

We are going to condition on what subsets the endpoints belong to, and count the # of edges.

- If both endpoints are in the set with $k$-vertices, there are $\binom{k}{2}$ edges.
- If both endpoints are in the set with $(n-k)$-vertices, there are $\binom{n-k}{2}$ edges.
- If there is one endpoint in each subset, there are $k(n-k)$ edges, since each of the $k$ vertices in the 1st subset must connect to the $n-k$ vertices in the 2nd subset.

Because each edges exists in only one of these conditions, the number of edges in $K_n$ is $\binom{k}{2} + k(n-k) + \binom{n-k}{2}$

Thus, $\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}$. ∎

4) (12 points) Page 248 #2. Improve Theorem 6.2.1 by proving that any tree has at least $\Delta$ leaves, where $\Delta$ is the maximum degree in the graph. Also, for each $n \geq 2$, give an example of a graph with exactly $\Delta$ leaves, where $2 \leq \Delta < n$.

Thrm 6.2.1: If T is a tree w/ at least two vertices, then T has at least two leaves.

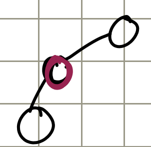
Pf// Let T be a tree with maximum degree $\Delta$. Let v be a vertex in T where $\deg(v) = \Delta$. Since T is a tree, it must be connected and acyclic. If we were to remove v from T, we would then have $\Delta$ subtrees, because there are $\Delta$ connections to v. Because T is acyclic, we know that it must form $\Delta$ subtrees because none of the remaining edges can cycle.

Since T was acyclic, each subtree must be acyclic, so each subtree must have at least one leaf. Therefore, there are at least $\Delta$ leaves in T. ∎

Examples:

$3 \geq 2$ $\quad 2 \leq 2 < 3$

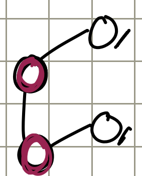
$n = 3$ $\quad \Delta = 2$



2 leaves

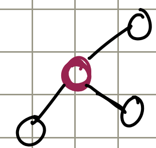
$4 \geq 2$ $\quad 2 \leq 2 < 4$

$n = 4$ $\quad \Delta = 2$



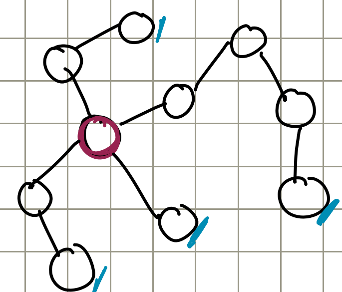
2 leaves

$n = 4$ $\quad \Delta = 3$



3 leaves

$n = 10$ $\quad \Delta = 4$



4 leaves

To construct a tree for any $n \geq 2$ with exactly $\Delta$ leaves where $2 \leq \Delta < n$

Start with a vertex v of degree $\Delta$, so there will be $\Delta$ edges.

We now need to determine how many vertices should be attached to each edge on v.

Each edge should contain a path of $m_i$ vertices, where $\sum_{i=1}^{\Delta} m_i = n-1$.

This will create a tree with $n \geq 2$ verticies with exactly $\Delta$ leaves where $2 \leq \Delta < n$.

5) Page 248 #3 Prove Theorem 6.2.4. Here's how to do this. Prove that

1. (a) implies (b)
2. (b) implies (c)
3. (c) implies (d)
4. (d) implies (a)

**Theorem 6.2.4:** If $T$ is a graph on $n$ vertices, then the following statements are equivalent.

a. $T$ is a tree.
b. $T$ is connected and has $n-1$ edges.
c. $T$ is acyclic and has $n-1$ edges.
d. $T$ is connected, but the deletion of any edge disconnects the graph.

Once you have done this, then any of these imply any of the others.

1) $a \Rightarrow b$: If $T$ is a tree, then $T$ is connected and has $n-1$ edges.

Pf. By definition, because $T$ is a tree, then it must be connected. To prove that $T$ has $n-1$ edges, we can restate the proof for Thm. 6.2.3. ■

2) $b \Rightarrow c$: If $T$ is connected and has $n-1$ edges, then $T$ is acyclic and has $n-1$ edges

Pf. By the given, we know that $T$ has $n-1$ edges. So, we just need to show that $T$ is acyclic.

Assume that $T$ contains a cycle, $C$. Let $e$ be an edge that is a part of $C$. We will remove $e$ from $T$ to create a new tree, $T-e$. Now $T-e$ is still connected because we can still reach all vertices by alternate edges.

$T-e$ has $n$ vertices, but $(n-1)-1 = n-2$ edges, which leads us to a contradiction. Since $T-e$ is connected, it must have $n-1$ edges.
Thus, $T$ must be acyclic. ■

3) $c \Rightarrow d$: If $T$ is acyclic and has $n-1$ edges, then $T$ is connected, but the deletion of any edge disconnects the graph.

Pf. First, we need to show that $T$ is connected. Assume $T$ is not connected. Then $T$ must consist of connected components. Since $T$ is not connected, it must have at least 2 connected components. Because $T$ is acyclic, each component is also acyclic. So, each componenent must be a tree by definition.

We have the following components: $C_1, C_2, \ldots, C_k$ where $k \geq 2$. Because $T$ has $n-1$ edges, then, $\sum_{i=1}^{k} e(C_i) = n-1$, or the sum of all edges of $C_1, \ldots C_k$ must be $n-1$.

Let $n_i$ be the total # of vertices in $C_i$. So, we know than $\sum_{i=1}^{k} n_i = n$. Because each $C_i$ is a tree by definition, then $e(C_i) = n_i - 1$

$$\text{So } \sum_{i=1}^{k} e(C_i) = \sum_{i=1}^{k} n_i - 1 = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = n - k. \text{ So } \sum_{i=1}^{k} e(C_i) = n-k.$$

Now, $n-k = n-1$, so $k=1$, which is a contradiction, since we said $k \geq 2$.
Thus, $T$ must be connected.

Now, we must show that the deletion of any edge disconnects the graph.
Let $e$ be an edge that is a part of $T$. We know that $T$ has $n-1$ edges by the given, is acyclic, and by above is connected. Thus, $T$ is a tree. We will remove $e$ from $T$ to create a new tree, $T-e$.

Assume that if we remove e that it will still be a connected graph. We know that because T was acyclic, removing an edge will not create a cycle, so T-e is still acyclic. Because we Removed the edge, T-e still has all n vertices, but $(n-1)-1 = n-2$ edges, which leads us to a contradiction. Since T-e is connected by our Assumption, then it must have n-1 edges.
So, the deletion of any edge disconnects the graph.

So, $c \Rightarrow d$. ∎

4) $d \Rightarrow a$: If T is connected, but the deletion of any edge disconnects the graph, then T is a tree.

We need to prove that T is a tree, a connected, acyclic graph. By our hypothesis, T is connected. So, we just need to show that T is acyclic.

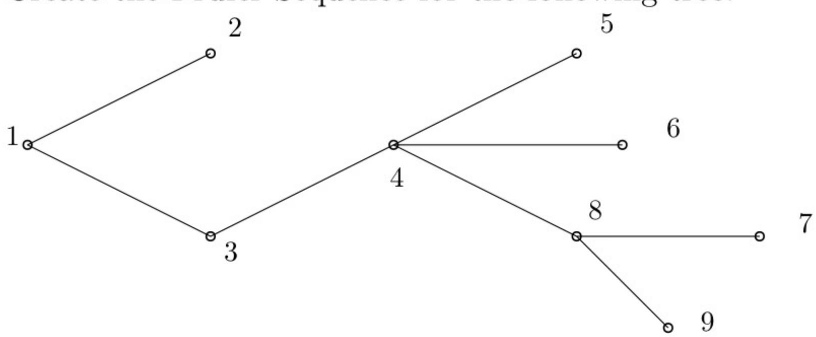
Assume that T contains a cycle, C. Let e be an edge that is a part of C. We will Remove e from T to create a new tree, T-e. Now T-e is still connected because We can still reach all vertices by alternate edges. However, this contradicts our hypothesis, and T must be acyclic.

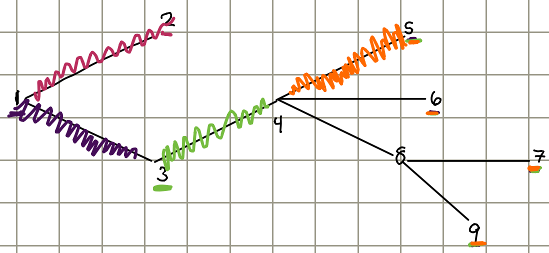
So T is a connected, acyclic graph, so T is a tree ∎

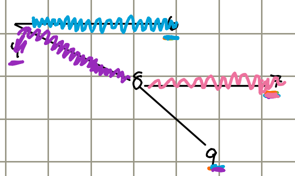
Since we showed $a \Rightarrow b \Rightarrow c \Rightarrow d \Rightarrow a$, We have proved Thrm. 6.2.4. ∎

# 6) a)

Create the Prüfer Sequence for the following tree.




L = ( )
L = (1)
L = (1, 3)
L = (1, 3, 4)
L = (1, 3, 4, 4)
L = (1, 3, 4, 4, 4)
L = (1, 3, 4, 4, 4, 8)
L = (1, 3, 4, 4, 4, 8, 8)

So, the Prüfer sequence of the tree is (1,3,4,4,4,8,8).

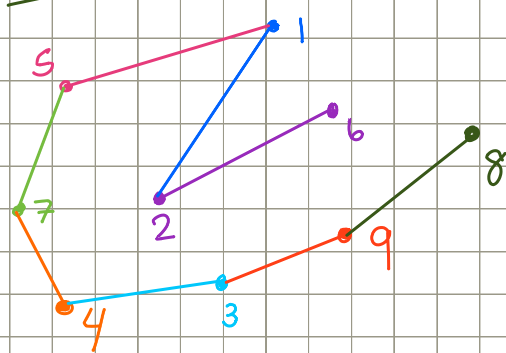
# b)

Create the tree with Prüfer sequence (2,1,5,7,4,3,9). 6,8 missing

[9]

| L | U | u | l |
|---|---|---|---|
| (2, 1, 5, 7, 4, 3, 9) | ( ) | 6 | 2 |
| (1, 5, 7, 4, 3, 9) | (6) | 2 | 1 |
| (5, 7, 4, 3, 9) | (6, 2) | 1 | 5 |
| (7, 4, 3, 9) | (6, 2, 1) | 5 | 7 |
| (4, 3, 9) | (6, 2, 1, 5) | 7 | 4 |
| (3, 9) | (6, 2, 1, 5, 7) | 4 | 3 |
| (9) | (6, 2, 1, 5, 7, 4) | 3 | 9 |
| ( ) | (6, 2, 1, 5, 7, 4, 3) | | |

Missing 8 and 9



7) (12 points) Page 248 #8 Here's another way to prove Cayley's formula. Let $L(n,k)$ denote the number of labeled trees with vertex set $[n]$ and where vertex $n$ has degree $k$.

**Theorem 6.2.5 (Cayley):** For $n \geq 2$, there are $n^{n-2}$ labeled trees on $n$ vertices.

a) Use properties of multinomial coefficients and Corollary 6.2.7 of htis section to prove that $L(n,k) = \binom{n-2}{k-1}(n-1)^{n-k-1}$.

Pf: We know that $L(n,k)$ is the # of labeled trees with vertex set $[n]$ and vertex $n$ has degree $k$.

So, using Col 6.7.2, we know $\binom{n-2}{d_1-1, d_2-1, \cdots, \underbrace{d_n-1}_{=k}} = \binom{n-2}{d_1-1, d_2-1, \cdots, k-1} = \frac{(n-2)!}{(d_1-1)!\,(d_2-1)!\cdots(k-1)!}$

=k because vertex $n$ has degree $k$.

We know that the sum of all degrees is $2e(G)$, so $\sum_{i=1}^{n} d_i = 2(n-1)$. Because we know that $d_n$ has a degree $k$, $\sum_{i=1}^{n-1} d_i = 2(n-1) - k$

**Theorem 4.1.3:** for $n > 0$ and $k > 0$, $k^n = \sum_{t_1+t_2+\cdots t_k = n} \binom{n}{t_1, t_2, \ldots, t_k}$.

To get the $(n-1)^{n-k-1}$ part of the equation, we need to use thrm 4.1.3, so we need to find $\sum_{i=1}^{n-1} d_i - 1 = \left(\sum_{i=1}^{n-1} d_i\right) - (n-1) = 2(n-1) - k - (n-1) = n-1-k = n-1-k.$

Now, to find $L(n,k)$, we need to sum over all possible $(d_1-1, d_2-1, \ldots, d_n-1)$

$$L(n,k) = \sum_{d_1-1+d_2-1+\cdots d_{n-1}-1 = n-k-1} \frac{(n-2)!}{(d_1-1)!\,(d_2-1)!\cdots(d_{n-1}-1)!\,(k-1)!}$$

$$= \frac{(n-2)!}{(k-1)!} \sum_{d_1-1+d_2-1+\cdots+d_{n-1}-1 = n-k-1} \frac{1}{(d_1-1)!\,(d_2-1)!\cdots(d_{n-1}-1)!} \cdot \frac{(n-k-1)!}{(n-k-1)!}$$

Thrm 4.1.3 $(n-1)^{n-k-1} = \sum_{d_1-1+d_2-1+\cdots+d_{n-1}-1=n-k-1} \frac{(n-k-1)!}{(d_1-1)!\,(d_2-1)!\cdots(d_{n-1}-1)!}$

$$= \boxed{\frac{(n-2)!}{(k-1)!} \cdot \frac{(n-1)^{n-k-1}}{(n-k-1)!}}$$

$\binom{n-2}{k-1} = \frac{(n-2)!}{(k-1)!\,((n-2)-(k-1))!}$

$(n-2)-(k-1) = n-2-k+1 = n-k-1$

So, $L(n,k) = \binom{n-2}{k-1}(n-1)^{n-k-1}$ ■

**b)** Derive Cayley's formula by summing $L(n,k)$ over appropriate values of $k$.

$$L(n,k) = \binom{n-2}{k-1}(n-1)^{n-k-1}$$ using the hint in the book $\sum_{k=1}^{n-1}$

$$\sum_{k=1}^{n-1} L(n,k) = \sum_{k=1}^{n-1}\binom{n-2}{k-1}(n-1)^{n-k-1}$$ Let $j = k-1$

$k = j+1$

$1 = j+1$, $j=0$ $\quad$ $n-1 = j+1$, $n-2 = j$

$$= \sum_{j=0}^{n-2}\binom{n-2}{j}(n-1)^{n-(j+1)-1}$$

$$= \sum_{j=0}^{n-2}\binom{n-2}{j}(n-1)^{n-j-2}$$ < binomial Thm.

Let $i = n-j-2$ $\quad$ $j = n-2-i$

$i = n-j-2$, $j=0$, $i=n-2$ $\quad$ $j=n-2$, $i = n-n+2-2 = 0$

$$\sum_{k\geq 0}\binom{\cancel{n}\ n-2}{k}\cancel{x}^k_{\ n-1} = (1+\cancel{x})^{\cancel{n}\ n-2}_{(n-1)}$$

$$= \sum_{i=0}^{n-2}\binom{n-2}{i}(n-1)^{i}$$

$$= (1+(n-1))^{n-2}$$ using the binomial Thm.

$$= n^{n-2}$$, which is Cayley's formula. ∎

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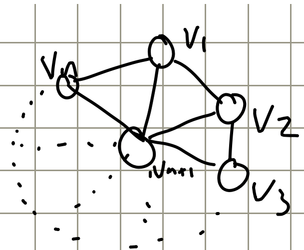
**8)** (8 points) Page 259 #4 Determine, with proof, $\chi(W_n)$. Here, the graph $W_n$ is the "wheel" consisting of a cycle $C_n$ as well as one additional vertex that is adjacent to every vertex on the cycle. (So $W_n$ has $n+1$ vertices.




Pf To find $\chi(W_n)$, we need to examine 2 cases, based on $n$ being odd or even.

Case 1: $n$ is even

Using our Proof from 6.3.V.1, we know that if $n$ is even, $\chi(C_n) = 2$. However, since vertex $V_{n+1}$ as shown above is adjacent to every vertex in $C_n$, we need to make that a different color than the 2-coloring from the cycle. So, $\chi(W_n) = 3$, when $n$ is even.

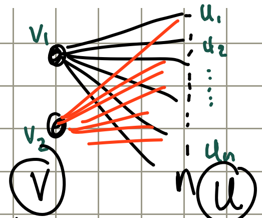
Case 2: $n$ is odd

Using our Proof from 6.3.V.1, we know that if $n$ is odd, $\chi(C_n) = 3$. However, since vertex $V_{n+1}$ as shown above is adjacent to every vertex in $C_n$, we need to make that a different color than the 3-coloring from the cycle. So, $\chi(W_n) = 4$, when $n$ is odd.

So, $\chi(W_n) = \begin{cases} 3 & \text{when } n \text{ is even} \\ 4 & \text{when } n \text{ is odd.} \end{cases}$ ∎

9) (10 points) Page 260 #9bc Determine the chromatic polynomial of each of the following graphs.

b) $K_{2,n}$




$V = \{v_1, v_2\}$
$U = \{u_1, u_2, \ldots, u_n\}$

**Case 1:** $v_1$ and $v_2$ are the same color

$v_1$ and $v_2$ have $k$ ways to choose a color. Since the color in each vertex in $U$ must be different than the color chosen for $v_1$ and $v_2$, there are $k-1$ choices for the color for each vertex.

So, there are $k(k-1)^n$ colorings for this case.

**Case 2:** $v_1$ and $v_2$ are different colors.

There are $k$ ways to choose a color for $v_1$ and $k-1$ ways for $v_2$. Since the color in each vertex in $U$ must be different than the color chosen for $v_1$ and $v_2$, there are $k-2$ choices for the color for each vertex.

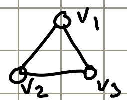
So, there are $k(k-1)(k-2)^n$ colorings for this case.

So, $p(K_{2,n}, k) = k(k-1)^n + k(k-1)(k-2)^n$

c) $C_3 \cup P_4 \cup K_5$

We know that $p(C_3 \cup P_4 \cup K_5, k) = p(C_3, k) \cdot p(P_4, k) \cdot p(K_5, k)$. So, we will examine each graph separately:

- $C_3$




For $C_3$, we will need 3 colors,
- $v_1$ has $k$ choices
- $v_2$ has $k-1$ choices
- $v_3$ has $k-2$ choices.

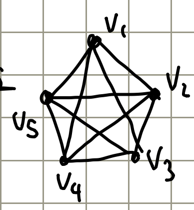
Thus, there are $k(k-1)(k-2)$ colorings.

- $P_4$


For $P_4$, we start by choosing a color for $v_1$
- $v_1$ has $k$ choices
- $v_2$ has $k-1$ choices
- $v_3$ has $k-1$ choices
- $v_4$ has $k-1$ choices

Thus, there are $k(k-1)^3$ colorings

• $K_5$




For $K_5$, we will need 5 colors:

- $v_1$ has $k$ choices
- $v_2$ has $k-1$ choices
- $v_3$ has $k-2$ choices.
- $v_4$ has $k-3$ choices
- $v_5$ has $k-4$ choices.

Thus, there are $\underline{k(k-1)(k-2)(k-3)(k-4)}$ colorings.

$$\text{So } p(C_3 \cup P_4 \cup K_5, k) = \underline{p(C_3,k)} \cdot \underline{p(P_4,k)} \cdot \underline{p(K_5,k)}$$

$$= k(k-1)(k-2) \cdot k(k-1)^3 \cdot k(k-1)(k-2)(k-3)(k-4)$$

$$p(C_3 \cup P_4 \cup K_5, k) = k^3 (k-1)^5 (k-2)^2 (k-3)(k-4)$$