1) Prove $A \subseteq \mathbb{R}^2$ is compact iff A is closed + bounded.

($\Rightarrow$) Assume A is compact.

Closed

Assume A is not closed.

Let p be a l.p. of A s.t $p \notin A$.

I look up that notation to avoid $(x-a)^2+(y-b)^2<(\frac{1}{n})^2$ s.t $p=(a,b)$ + $n \in \mathbb{N}$

Consider the open cover $\mathcal{C}_1$ of A which consists of all open balls centered at p, with radius $\frac{1}{n}$, or $\mathcal{C}_1 = \{B(p,\frac{1}{n}) \mid n \in \mathbb{N}\}$.

Since A is compact, $\exists$ a finite subcover of $\mathcal{C}_1$. Thus, $\exists$ N s.t.

$A \subseteq B(p,\frac{1}{N})$. So, $B(p,\frac{1}{N}) \cap A = \emptyset$, so p is not a L.P of A $\rightarrow\leftarrow$

So A must be closed.

Bounded

Consider the open cover $\mathcal{C}_2 = \{(-n,n)\times(-m,m) \mid n,m \in \mathbb{N}\}$ of A.

Since A is compact, $\exists$ a finite subcover of $\mathcal{C}_2$. Thus, $\exists N, M \in \mathbb{N}$ s.t $\forall (x,y) \in A$, $-N<x<N$ and $-M<y<M$. So, A is bounded.

($\Leftarrow$) Assume A is closed + bounded.

Since A is bounded, $\exists n,m \in \mathbb{N}$ s.t. $A \subseteq [-n,n]\times[-m,m]$.

We know that $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$, and $[-n,n] \subseteq \mathbb{R}$ and $[-m,m] \subseteq \mathbb{R}$.

By Thrm. 8.2.2, since $[-n,n]$ is closed + bounded, it is compact in $\mathbb{R}$. Same logic applies for $[-m,m]$.

By Thrm 9.1.2, since $[-n,n]$ + $[-m,m]$ are both compact, then

$[-n,n]\times[m,m] \stackrel{=X}{}$ is compact. Since $A \subseteq X$, and because Every closed subspace of a compact space is compact, A is compact.

I found a Thrm from a lecture from UPenn Math to help.

2) Prove $A = \{x \mid f(x) < g(x)\}$ is open in X.

Let $x \in A$, so $f(x) < g(x)$. So, $\exists\ a, b \in Y$ s.t. $f(x) < a < b < g(x)$.

Now, $\exists$ an open subset U of Y s.t. $U = (f(x), a)$ and so, $f(x) \in U$.

Similarly, $\exists$ open set $V \subseteq Y$ s.t $V = (b, g(x))$ and so $g(x) \in V$. Now, let $S = f^{-1}(U)$

and let $W = g^{-1}(V)$. s.t $S, W \subseteq X$. By Thrm. 9.2.1 ②, Both S and W are open in X.

Thus, $x \in W$ and $x \in S$. Thus, $x \in W \cap S$, and $W \cap S$ is open in X.

Now, we need to show that $W \cap S \subseteq A$ prov~~ing~~ A is open.

Let $p \in W \cap S$. We are trying to show that $f(p) < g(p)$.

Thus, $p \in W$ and $p \in S$. So $f(p) \in U$ and $g(p) \in V$.

By the def of $U + V$, $f(x) < f(p) < a < b < g(p) < g(x)$, so $f(p) < g(p)$

Thus. $W \cap S \subseteq A$ and so A is open in X. ■ :)

3) Prove $x \in X$, $f(x) = g(x)$.

Let X be a topological subset, with the Dense subset D. Y is Hausdorff space.

$f: X \to Y$, $g: X \to Y$

for $\forall\ d \in D$, $f(d) = g(d)$

Pf/ Suppose $\exists\ x \in X$ s.t $f(x) \neq g(x)$. Then because Y is Hausdorff,

$\exists$ disjoint open sets U, V in Y s.t. $f(x) \in U$ and $g(x) \in V$.

Now, since f is continuous, $f^{-1}(U)$ is open in X, and similarly w/ g,

$g^{-1}(V)$ is open in X. So, because $x \in f^{-1}(U)$ and $x \in g^{-1}(V)$

if $f^{-1}(U) \cap g^{-1}(V) = W$, $x \in W$, which is open in X.

Since D is dense, $\exists\ d \in D$ s.t $d \in W$. Now, $f(d) \in U$, and since $f(d) = g(d)$ + $g(d) \in V$

then $f(d) \in V$. $\rightarrow\leftarrow$

This is a contradiction, because we said $U + V$ we disjoint.

Thus, For any $x \in X$, $f(x) = g(x)$.

■ :)

4) Let X, Y, Z be topological space. $f: X \to Y$ and $g: Y \to Z$ are contin.

Prove $(g \circ f): X \to Z$ is continuous.

Pf. Let $W \subseteq Z$ be an open subset. Show $(g \circ f)^{-1}(W)$ is open in X.

Because g is continuous, $g^{-1}(W)$ is open in Y.

Let $V = g^{-1}(W)$.

Since f is continuous, $f^{-1}(V)$ is open in X.

Now, lets look at

$(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)) = f^{-1}(V)$, which we know is open in X,

so, $g \circ f: X \to Z$ is continuous $\blacksquare$