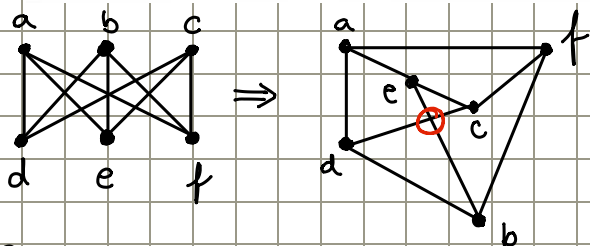


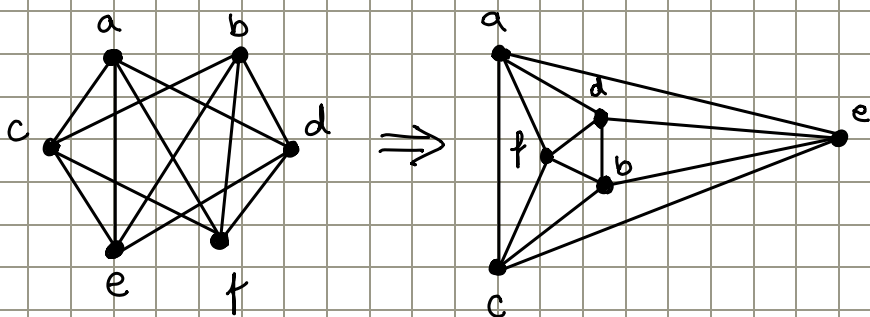
# Homework # 7 - Math 590

1) (5 points) (p.137 #5.49) Determine  $cr(K_{3,3})$  without using Theorem 5.26.



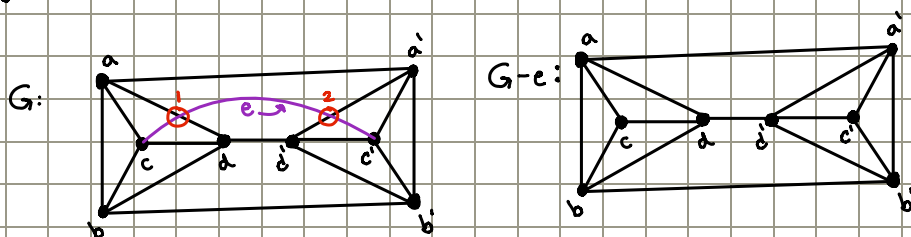
Because  $K_{3,3}$  is non-planar  $cr(K_{3,3}) \geq 1$ . Additionally, I was able to draw  $K_{3,3}$  with one crossing, so  $cr(K_{3,3}) \leq 1$ . Thus,  $cr(K_{3,3}) = 1$ .

2) (5 points) (p.137 #5.51) Determine  $cr(K_{2,2,2})$ .



I was able to draw  $K_{2,2,2}$  as a planar graph, so  $cr(K_{2,2,2}) = 0$ .

3) (5 points) (p.137 #5.55) Disprove: If  $G$  is a nonplanar graph containing an edge  $e$  such that  $G - e$  is planar, then  $cr(G) = 1$ .



Clearly  $G$  is non-planar, and  $G - e$  is planar. However  $cr(G) = 2$ . If edge  $e = cc'$ ,  $e$  first needs to leave the interior of the first  $K_4$ , crossing 1. Then  $e$  needs to cross again, to get into the interior of the second  $K_4$ , crossing 2. So  $cr(G) = 2$ .

4) (10 points) (p.166 #6.2) Prove that the size of every  $k$ -chromatic graph is at least  $\binom{k}{2}$ .  $m \geq \frac{(k-1)k}{2}$

Prf: Let  $G$  be a graph of order  $n$  and size  $m$ . Let  $G$  be  $k$ -chromatic, where  $k \geq 2$ . Because  $G$  is  $k$ -chromatic, there is no  $k-1$  coloring, since  $k$  is the minimum. So, we can label the colors  $1, 2, \dots, k$ . Now, for any 2 distinct color classes, where  $1 \leq i < j \leq k$ . Now, we know there must be at least one edge between color class  $i$  and color class  $j$  or we could combine these two color classes into one, which would give a  $k-1$  coloring, which is a contradiction.

So, there must be at least  $\binom{k}{2}$  edges connecting color class  $k$  to each of the  $k-1$  other classes. Thus,  $m \geq \binom{k}{2}$ . ■

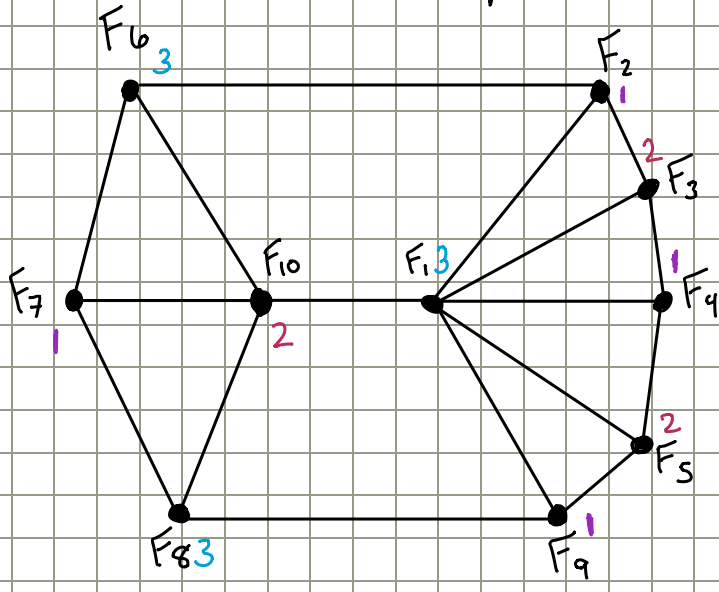
5) (12 points) Two dentists are having new offices designed for themselves. In the common waiting room for their patients, they have decided to have an aquatic area containing fish tanks. Because some fish require a cold water environment while others are more tropical and because some fish are aggressive with other types of fish, not all fish can be placed in a single tank. It is decided to have ten exotic fish, denoted by  $F_1, F_2, \dots, F_{10}$ , where the fish that cannot be placed in the same tank as  $F_i$  ( $1 \leq i \leq 10$ ) are listed to the right of

- |            |                              |         |                    |         |                    |
|------------|------------------------------|---------|--------------------|---------|--------------------|
| $F_1$ :    | $F_2, F_3, F_4, F_5, F_{10}$ | $F_2$ : | $F_1, F_3, F_6$    | $F_3$ : | $F_1, F_2, F_4$    |
| $F_4$ :    | $F_1, F_3, F_5$              | $F_5$ : | $F_1, F_4, F_9$    | $F_6$ : | $F_2, F_7, F_{10}$ |
| $F_7$ :    | $F_6, F_8, F_{10}$           | $F_8$ : | $F_7, F_9, F_{10}$ | $F_9$ : | $F_5, F_8, F_{10}$ |
| $F_{10}$ : | $F_1, F_6, F_7, F_8, F_9$    |         |                    |         |                    |

What is the minimum number of fish tanks required?

I started by Drawing each fish as a vertex, with an edge to each fish it cannot be in the same tank as. Thus, if we find a chromatic number, we can know how many tanks.

First, by Thrm. 6.4,  $\chi(G) \leq 6$ , since  $\Delta(G) = 6$ . Since there is a  $K_3$  subgraph,  $\chi(G) \geq 3$ . I was able to find a 3-coloring, so  $\chi(G) \leq 3$ . Since  $3 \leq \chi(G) \leq 3$ ,  $\chi(G) = 3$ .



Tank 1:  $F_2, F_4, F_7, F_9$

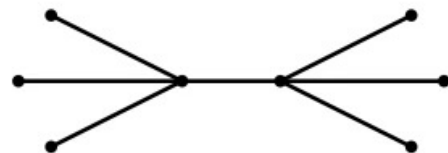
Tank 2:  $F_3, F_5, F_{10}$

Tank 3:  $F_1, F_6, F_8$

6) (12 points) (p.167 #6.10) For the double star  $T$  containing two vertices of degree 4 shown below, what upper bound on  $\chi(T)$  are given by (a) Thm 6.2, (b) Thm 6.3 and (c) Brooks theorem (Thm. 6.4)? (Note one of the answers in the book is incorrect).

a) Thm 6.2: For every graph  $G$  with degree sequence  $d_1 \geq d_2 \geq \dots \geq d_n$ , we have

$$\chi(G) \leq \max_{1 \leq i \leq n} \min\{i, d_i + 1\}$$



This graph has degree sequence  $4, 4, 1, 1, 1, 1, 1, 1$ ; and  $n = 8$

$i$	$d_i + 1$	$\min\{i, d_i + 1\}$
1	5	1
2	5	2
3	2	2
4	2	2
5	2	2
6	2	2
7	2	2
8	2	2

Thus,  $\chi(T) \leq 2$ .

$\max_{1 \leq i \leq n} = 2$

b) Thm 6.3: For every graph  $G$ ,  $\chi(G) \leq 1 + \max_{H \subseteq G} \delta(H)$ . (Where the maximum is taken over all subgraphs  $H$  of  $G$ ).

Since  $T$  is a tree, every subgraph will either be a smaller tree, or a forest. Additionally, any subgraph will either have a leaf, or a disconnected vertex.

If there are no disconnected vertices in the subgraph,  $\delta(H) = 1$ , since by Lemma 1.1,  $H$  has at least 2 leaves.

If  $H$  does have a disconnected vertex,  $\delta(H) = 0$ .

So, the  $\max_{H \subseteq G} \delta(H) = \max\{0, 1\} = 1$ . Thus,  $\chi(T) \leq 1 + 1 \Rightarrow \chi(T) \leq 2$ .

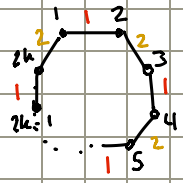
c) Thm 6.4 (Brooks Thm): For every connected graph  $G$  that is not an odd cycle or a complete graph,  $\chi(G) \leq \Delta(G)$ .

$T$  is clearly connected, and since it is a tree, it is acyclic and not complete.

$\Delta(T) = 4$ . Thus,  $\chi(T) \leq 4$ .

7) (10 points) (p.167 #6.12) Without using König's line coloring theorem (Thm 6.7), prove that even cycles are of class one, so they have edge-chromatic number 2.

Pf: Let  $C_{2k}$  be an even-cycle, where  $k \geq 1$ . Because it is a cycle,  $\Delta(C_{2k}) = 2$ , so by Thm 6.5,  $\chi'(C_{2k}) \leq 1 + 2 \Rightarrow \chi'(C_{2k}) \leq 3$ . Additionally, because  $\Delta(C_{2k}) = 2$ , if a vertex  $v$  has degree  $\Delta$ , the  $\Delta$  edges that are incident with  $v$  must all be different colors. Thus,  $\chi'(C_{2k}) \geq 2$ .



So  $2 \leq \chi'(C_{2k}) \leq 3$ . Now the vertices of  $C_{2k}$  are  $v_1, v_2, \dots, v_{2k}$  with edges  $e_i = v_i v_{i+1}$  s.t.  $1 \leq i \leq 2k$  and  $v_{2k+1} = v_1$ .

Now, there are  $2k$  edges of a cycle. Label all edges  $e_i$  s.t.  $i$  is odd with color 1 and edges  $e_i$  s.t.  $i$  is even with 2. Because there are an even number of edges, edge  $e_{2k}$  is labeled with 2, which is different than  $e_1$ , which is 1. So  $\chi'(C_{2k}) \leq 2$ .

Thus, since  $2 \leq \chi'(C_{2k}) \leq 2$ ,  $\chi'(C_{2k}) = 2$ , so since  $\chi'(C_{2k}) = \Delta(C_{2k})$ ,  $C_{2k}$  is class one. ■

8) (10 points) (p.167 #6.13) Prove that odd cycles are of class two, so they have edge-chromatic number 3.

Let  $C_{2k+1}$  be an odd-cycle, where  $k \geq 1$ . Because it is a cycle,  $\Delta(C_{2k+1}) = 2$ , so by Thm 6.5,  $\chi'(C_{2k+1}) \leq 1 + 2 \Rightarrow \chi'(C_{2k+1}) \leq 3$ . Additionally, because  $\Delta(C_{2k+1}) = 2$ , if a vertex  $v$  has degree  $\Delta$ , the  $\Delta$  edges that are incident with  $v$  must all be different colors. Thus,  $\chi'(C_{2k+1}) \geq 2$ .

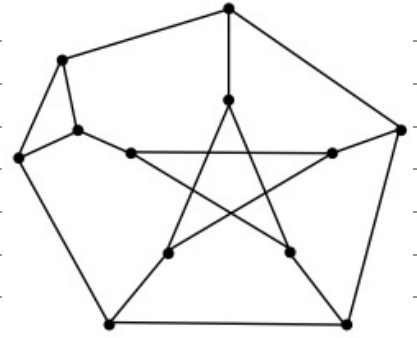
So  $2 \leq \chi'(C_{2k+1}) \leq 3$ . Now the vertices of  $C_{2k+1}$  are  $v_1, v_2, \dots, v_{2k}, v_{2k+1}$  with edges  $e_i = v_i v_{i+1}$  s.t.  $1 \leq i \leq 2k+1$  and  $v_{2k+2} = v_1$ .

For the odd cycle, we have  $2k+1$  edges. Begin with labeling  $e_i$  as 1 if  $i$  is odd and  $i \leq 2k$  and 2 if  $i$  is even. Now for  $e_{2k+1}$ , it's odd, but can't be labeled with 1 since  $e_{2k+1}$  is adjacent to  $e_1$ , so  $e_{2k+1}$  is labeled with 3. So there is no 2-edge-coloring for  $C_{2k+1}$ .

Thus,  $3 \leq \chi'(C_{2k+1}) \leq 3$ , so  $\chi'(C_{2k+1}) = 3 = 1 + \Delta(C_{2k+1})$ , and so  $C_{2k+1}$  is class two. ■

9)

(5 points) (p.167 #6.19) Determine the edge-chromatic number of the graph shown below.

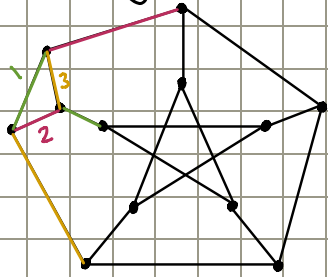


There is  $K_3$  subgraph, so  $\chi'(G) \geq 3$ . Also by Thrm. 6.5,  $\chi'(G) \leq 1 + \Delta(G) \Rightarrow \chi'(G) \leq 1 + 3 \Rightarrow \chi'(G) \leq 4$ . So  $3 \leq \chi'(G) \leq 4$ .

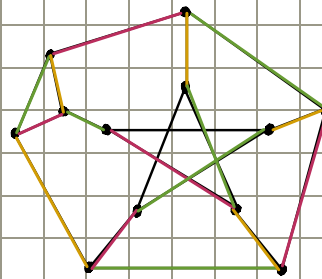
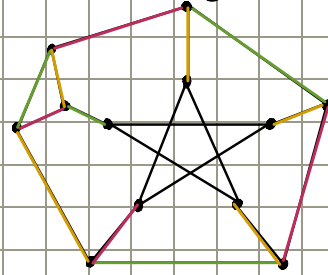
Now, I will show that there cannot be a 3-edge-coloring.

Starting with the 3-cycle, those must be 3 different colors.

Once you mark those, a few more edges get forced into a specific color.



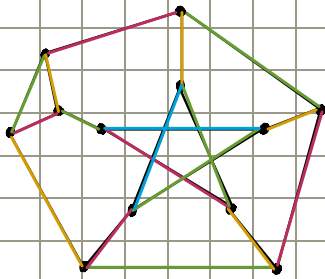
Continue coloring around the outer cycle.



This forces the edges to the star as specific colors.

Now at this point, there are two places where we need to introduce a 4<sup>th</sup> color.

So  $\chi'(G) = 4$ .



10) (10 points) (p.168 #6.23) Prove that the odd cycles are the only 3-critical graphs.

( $\Rightarrow$ ) { If  $G$  is an odd cycle, then it is 3-critical }

Let  $C_{2k+1}$  s.t.  $k \geq 1$  be an odd cycle. By 6.1 Notes  $\chi(C_{2k+1}) = 3$   
 For any vertex  $v \in V(C_{2k+1})$ ,  $C_{2k+1} - v$  would be a path, which means  $\chi(C_{2k+1} - v) = 2$ . So, since  $2 < 3$ ,  $C_{2k+1}$  is 3-critical.

( $\Leftarrow$ ) { If  $G$  is 3-critical, then it is an odd cycle. }

Let  $G$  be 3-critical. This means that  $G$  is 3-chromatic, so there is no 2 coloring, so there must be an odd cycle.

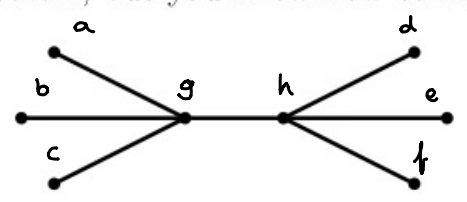
Let  $C$  be the shortest odd cycle. Suppose there are 2 vertices  $v_i, v_j \in V(C)$  s.t.  $v_i$  is not adjacent to  $v_j$  and  $v_i v_j \in E(G)$ . However, this would mean there is a shorter odd cycle, as  $C$  would split into 2 smaller cycles, one odd, one even. But this is a contradiction, since  $C$  was the smallest odd cycle. So,  $C$  is an induced odd cycle.

Now, let  $u \in V(G)$  but  $u \notin V(C)$ . By the def. of 3-criticality,  $\chi(G - u) = 2$ .  
 But  $G - u$  still contains all of  $C$ , and  $C$  is an odd cycle, so  $\chi(G - u) = 3$ .  
 So,  $V(G) = V(C)$ . Thus  $G = C$ , and  $G$  is an odd cycle.

Thus, odd cycles are the only 3-critical graphs. ■

11) (10 points) (p.168 #6.31) For the double star  $T$  of Exercise 6.10 (#6), what upper bound on  $\chi(T)$  is given by Theorem 6.19. (We did not discuss this theorem, but you know how to find everything needed to use it.)

Thrm. 6.19: For every graph  $G$  of order  $n$ ,

$$\chi(G) \leq \frac{\omega(G) + n + 1 - \alpha(G)}{2}$$


$\omega(G) = 2 \rightarrow$  no cycles, so largest complete graph is  $C_2$   
 $\alpha(G) = 6 \rightarrow \{a, b, c, d, e, f\}$   
 $n = 8$

$$\chi(G) \leq \frac{\omega(G) + n + 1 - \alpha(G)}{2} = \frac{2 + 8 + 1 - 6}{2} = \frac{5}{2} \Rightarrow \chi(G) \leq \frac{5}{2} \Rightarrow \chi(G) \leq 2$$

12) (10 points) Determine whether  $\overline{C_8}$  is perfect.

By Thrm. 6.20  $\overline{C_8}$  is perfect iff  $C_8$  is perfect.

Now,  $C_8$  is perfect if  $\chi(H) = \omega(H)$  for every induced subgraph  $H$  of  $C_8$ .

Case 1:  $H = C_8$

↳  $\chi(H) = 2$  since it is an even cycle.

$\omega(H) = 2$  since the only complete subgraph is  $K_2$ .

$$\chi(H) = \omega(H) \checkmark$$

Case 2:  $H$  is a connected path

↳  $\chi(H) = 2$  since you can alternate vertex colors.

$\omega(H) = 2$  since the only complete subgraph is  $K_2$ .

$$\chi(H) = \omega(H) \checkmark$$

Case 3:  $H$  is an isolated vertex

↳  $\chi(H) = 1$  since you only need one color.

$\omega(H) = 1$  since the only complete subgraph is  $K_1$ .

$$\chi(H) = \omega(H) \checkmark$$

Every induced subgraph is a disjoint union of the cases, so  $\chi(H) = \omega(H)$ .

So  $C_8$  is perfect and thus  $\overline{C_8}$  is perfect.